GROWTH AND APPROXIMATION OF ANALYTIC FUNCTIONS AND SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
ATUL NAUTIYAL

to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

DECEMBER, 1981

CERTIFICATE

This is to certify that the work embodied in the thesis entitled GROWTH AND APPROXIMATION OF ANALYTIC FUNCTIONS AND SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS being submitted by Atul Nautiyal has been carried out under my supervision and that it has not been submitted elsewhere for the award of any degree or diploma.

December 1981

(G.P. Kapoor)
Assistant Professor
Department of Mathematics
Indian Institute of Technology, Kanpur.

ACKNOWLEDGEMENTS

I take this opportunity to express my deep sense of gratitude to my thesis supervisor Dr. G.P. Kapoor for suggesting the problems and his valuable guidance in the course of present work. I shall ever remain indebted to him for his helpful attitude, constant encouragement, constructive criticisms and useful suggestions.

I wish to express my thanks to my teachers Professor O.P. Juneja, Professor T. Pati and Professor R.S.L. Srivastava who had always been the source of inspiration to me. I am also thankful to Dr. Prem Singh for his kind interest in the present work.

I gratefully acknowledge the financial assistance from Indian Institute of Technology, Kanpur.

The careful and efficient stencil cutting of Mr. G.L. Misra and Mr. S.K. Tewari is to be highly appreciated. Finally, I extend my thanks to Mr. A.N. Upadhyaya for neatly cyclostyling the thesis.

Atul Nautiyal

December 1981

(Atul Nautiyal)
Department of Mathematics
Indian Institute of Technology, Kanpur.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. The theory of approximation of functions originated in th classical approximation theorem of Weierstrass. Chebyshev's concept of best approximation in uniform norm and the converse theorem of Bernstein put it on a sound footing. Runge's theorem that if $\mathcal{D}$ is a union of countably many disjoint simply connectec domains, none of which contains the point at infinity, then any analytic function on $\mathcal{D}$ can be uniformly approximated by polynomials on compact subsets of $\mathcal{D}$, aroused interest in the approximation of analytic functions by polynomials on subsets of the complex plane. Significant contributions in this direction were made by Faber, Keldysh, Walsh and others till Mergelyan came up with his epoch making result that, if $\mathcal{E}$ is a compact set, then a necessary and sufficient condition that every function continuous on $\mathcal{E}$ and analytic in the interior points of $\mathcal{E}$ can be uniformly approximated on $\mathcal{E}$ by polynomials is that $\mathcal{E}$ does not separate the plane. Since then, the theory has been vastly enriched in different directions by a number of workers.

It was observed in early stages of development of the theory that the order of the magnitude of the minimal error in approximating a function $f(z)$ by polynomials of degree n is related to the differentiability properties of $f(z)$. The more differentiable

the function, the better it is suited for approximation. Furth- more, the degree of approximation also depends on the region o analyticity of the function in question. Thus, amongst the cla of analytic functions, the entire functions happen to occupy a unique and privileged position that they are best characterized by the 'rapidity' of convergence of their degree of approximati Further, various workers have studied the growth of an entire function in terms of its degree of approximation. For function analytic in a finite region also, some efforts have been made tc connect their growth with the degree of approximation.

Lately, there have been some attempts to connect the growth of entire solutions of certain partial differential equations with their degree of approximation though much is left desired. However, it seems that so far the interrelations between the growth of the solutions of partial differential equations which are regular in a finite region and their degree of approximation have not been investigated.

In the following sections we review some results in function theory that concern the growth and approximation of analytic functions and solutions of certain partial differential equations and are relevant to the present study.

1.2. Let $f(z)$ be an entire function of the complex variable $z = re^{i\theta}$ and let

$$M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|.$$

The function $M(r,f)$ is called the maximum modulus of $f(z)$ for $|z| = r$. Blumenthal [14] has shown that $M(r,f)$ is a steadily increasing function of r and that it is differentiable in adjacent intervals. Further, by Hadamard's three-circles theorem, it follows that $\log M(r,f)$ is a convex function of $\log r$.

An entire function $f(z)$ is said to be of finite order if there exists a constant A such that

$$(1.2.1) \qquad M(r,f) < \exp\,(r^A)$$

for all sufficiently large values of r. The greatest lower bound $\rho_\infty(f)$ of all such numbers A is called the order of the function $f(z)$. Thus,

$$(1.2.2) \qquad \rho_\infty \equiv \rho_\infty(f) = \limsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r} .$$

If no constant A can be found such that (1.2.1) holds, then $f(z)$ is said to be of infinite order and such functions are said to be of fast growth. The entire functions of zero order are said to be of slow growth.

For a more precise specification of the rate of growth of $f(z)$ the concept of type has been introduced. Thus, an entire function $f(z)$, of nonzero finite order $\rho_\infty(f)$ is said to be of type $T_\infty(f)$ if

$$(1.2.3) \qquad T_\infty \equiv T_\infty(f) = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_\infty(f)}} .$$

In 1933, Whittaker [131] introduced the concept of lower order for an entire function. Thus, an entire function f(z) is said to be of lower order $\lambda_\infty(f)$ $(\lambda_\infty(f) \leq \rho_\infty(f))$ if

(1.2.4) $$\lambda_\infty \equiv \lambda_\infty(f) = \liminf_{r \to \infty} \frac{\log \log M(r,f)}{\log r} .$$

An entire function f(z) is said to be of regular growth if $\rho_\infty(f) = \lambda_\infty(f)$ and is said to be irregular growth if $\lambda_\infty(f) < \rho_\infty(f)$.

In analogy with lower order, Shah [101] has introduced the concept of lower type $t_\infty(f)$ of an entire function f(z) of order $\rho_\infty(f)$ $(0 < \rho_\infty(f) < \infty)$ as

(1.2.5) $$t_\infty \equiv t_\infty(f) = \liminf_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_\infty(f)}} .$$

Since the entire function f(z) is analytic everywhere in the finite complex plane, it can be expanded in a Taylor series around any point z_o in the complex plane. However, without loss of generality we can take $z_o = 0$. Then f(z) has the representation

(1.2.6) $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients a_n's are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} [f(z)/z^{n+1}]\, dz = f^n(0)/n!$$

$f^n(0)$ being the value of the nth derivative of f(z) at z = 0.

For an entire function f(z) to be of finite order and finite type, necessary and sufficient conditions, in terms of its Taylor coefficients a_n, have been found [15, pp. 9-12]. Thus, the entire function f(z), given by (1.2.6), is of finite order $\rho_\infty(f)$, if and only if,

$$(1.2.7) \qquad \rho_\infty(f) = \limsup_{n \to \infty} \frac{n \log n}{-\log |a_n|} < \infty.$$

Further, the entire function f(z), given by (1.2.6), is of order $\rho_\infty(f)$ $(0 < \rho_\infty(f) < \infty)$ and type $T_\infty(f)$ $(0 < T_\infty(f) < \infty)$ if and only if,

$$(1.2.8) \qquad e\,\rho_\infty(f)\,T_\infty(f) = \limsup_{n \to \infty} n|a_n|^{\rho_\infty(f)/n}.$$

A formula analogous to (1.2.7) does not hold, in general, for lower order of an entire function f(z). Shah [96,97] has shown that if f(z), given by (1.2.6), is an entire function of order $\rho_\infty(f)$ and lower order $\lambda_\infty(f)$, then

$$(1.2.9) \qquad \lambda_\infty(f) \geq \liminf_{n \to \infty} \frac{n \log n}{-\log |a_n|} \geq \liminf_{n \to \infty} \frac{\log n}{\log |a_n/a_{n+1}|}$$

and

$$\rho_\infty(f) \leq \limsup_{n \to \infty} \frac{\log n}{\log |a_n/a_{n+1}|}.$$

Further, if

$$(1.2.10) \qquad \phi(n) = |a_n/a_{n+1}| \text{ is a nondecreasing function of } n \text{ for } n > n_o,$$

then

$$(1.2.11) \qquad \lambda_\infty(f) = \liminf_{n \to \infty} \frac{n \log n}{-\log |a_n|} = \liminf_{n \to \infty} \frac{\log n}{\log |a_n/a_{n+1}|}$$

and

$$\rho_\infty(f) = \lim_{n \to \infty} \sup \frac{\log n}{\log |a_n/a_{n+1}|} .$$

Juneja [42] and Juneja and Kapoor [44] have obtained coefficient formulae for the lower order $\lambda_\infty(f)$ of an entire funct given by (1.2.6), in which the nondecreasing nature of the function $\phi(n) = |a_n/a_{n+1}|$ is no more needed and so they hold fo every entire function. Thus, if f(z), given by (1.2.6), is an entire function of lower order $\lambda_\infty(f)$, then

(1.2.12) $$\lambda_\infty(f) = \max_{\{n_k\}} \{\lim_{k \to \infty} \inf \frac{n_k \log n_{k-1}}{-\log |a_{n_k}|}\},$$

(1.2.13) $$\lambda_\infty(f) = \max_{\{n_k\}} \{\lim_{k \to \infty} \inf \frac{(n_k - n_{k-1}) \log n_{k-1}}{\log |a_{n_{k-1}}/a_{n_k}|}\},$$

where maximum in (1.2.12) and (1.2.13) is taken over all increasing sequences $\{n_k\}$ of positive integers.

In the case of lower type also, a coefficient formula analogous to (1.2.8) does not hold. Shah [101] has shown that if f(z), given by (1.2.6), is an entire function of order $\rho_\infty(f)$ $(0 < \rho_\infty(f) < \infty)$ and lower type $t_\infty(f)$, then

(1.2.14) $$t_\infty(f) \geq \frac{1}{e\, \rho_\infty(f)} \lim_{n \to \infty} \inf n|a_n|^{\rho_\infty(f)/n}$$

and, further, if (1.2.10) holds, we have

(1.2.15) $$t_\infty(f) = \frac{1}{e\, \rho_\infty(f)} \lim_{n \to \infty} \inf n|a_n|^{\rho_\infty(f)/n} .$$

Some more results in this direction are found in Basinger [9], Clunie [20], Juneja [41], Juneja and Singh [48], Rahman [80], R.S.L. Srivastava [118], Srivastava and Singh [120], S.N. Srivastava [121] etc.

1.3. If the entire function f(z) is of infinite or zero order the definition of type is not feasible and so the growth of such functions can not be measured precisely by confining to the concept of order only. For studying the growth of such functions various attempts have been made by different workers, notable among them being those of Bajpai, Kapoor and Juneja [5], Iyer [40], Sato [89], Schönhage [90], Shah and Ishaq [103] etc. Recently, an elegant approach in this direction is made by Kapoor in [50] and is as follows.

For an entire function f(z), which is not a polynomial, set

$$\rho_\infty(p,q) = \limsup_{r \to \infty} \frac{\log_p M(r,f)}{\log_q r}$$

where p and q are integers such that, $p \geq q \geq 1$, $\log_0 x = x$, $\log_p x = \log(\log_{p-1} x)$, for $p \geq 1$. An entire function f(z) is said to be of index pair (p,q), $p \geq q \geq 1$ if $0 < \rho_\infty(p,q) < \infty$ and $\rho_\infty(p-1, q-1)$ is not a nonzero finite number. If $\rho_\infty(p,q)$ is never nonzero finite and $\rho_\infty(p,1) = 0$ for some p, then index pair of f(z) is defined as (m,1) where $m = \inf\{p : \rho_\infty(p,1) = 0\}$. If $\rho_\infty(p,q)$ is always infinite, index pair of f(z) is defined to be (∞,∞).

A scheme to compare the rates of growth of two entire functions $f_1(z)$ and $f_2(z)$ having index pairs (p_1,q_1) and (p_2,q_2), respectively, can be found in [45,50].

An entire function $f(z)$ is said to be of (p,q)-order $\rho_\infty(p,q)$ if $f(z)$ is of index pair (p,q), $p \geq q \geq 1$ and $b < \rho_\infty(p,q) < \infty$ where $b = 0$ or 1 according as $p > q$ or $p = q$. An entire function $f(z)$, of index pair (p,q), is said to be of lower (p,q)-order $\lambda_\infty(p,q)$ if

$$\lambda_\infty(p,q) = \liminf_{r \to \infty} \frac{\log_p M(r,f)}{\log_q r} .$$

An entire function $f(z)$ of (p,q)-order $\rho_\infty(p,q)$ is said to be of (p,q)-type $T_\infty(p,q)$ and lower (p,q)-type $t_\infty(p,q)$, if

$$\begin{matrix} T_\infty(p,q) \\ t_\infty(p,q) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log_{p-1} M(r,f)}{(\log_{q-1} r)^{\rho_\infty(p,q)}} .$$

Coefficient equivalents of (p,q)-order, lower (p,q)-order, (p,q)-type and lower (p,q)-type have been found in [45], [46] and [50]. Some more results concerning (p,q)-order etc. can be found in [50], [72,73] and [74].

Recently, Šeremeta [93] and Shah [102] have introduced the concepts of generalized orders of an entire function $f(z)$ in which more general functions than the iteration of logarithms are used to compare the growth of $\log M(r,f)$ with that of $\log r$. For this purpose, the following two classes Λ_* and L_*^0 are introduced.

A function $h(x)$ defined on $[a,\infty)$ is said to belong to the class L_*^o if on $[a,\infty)$ it is positive, differentiable, strictly increasing, tends to ∞ as $x \to \infty$ and satisfies the condition that

$$(1.3.1) \qquad \lim_{x \to \infty} \frac{h(x(1 + \delta(x)))}{h(x)} = 1,$$

for every function $\delta(x)$ such that $\delta(x) \to 0$ as $x \to \infty$. A function $h(x) \varepsilon L_*^o$ is said to belong to the class Λ_* if $h(x)$ is a slowly increasing function [91], i.e., if the following stronger condition holds in place of (1.3.1) ,

$$(1.3.2) \qquad \lim_{x \to \infty} \frac{h(cx)}{h(x)} = 1$$

for all c, $0 < c < \infty$. In the results stated in this section, wherever necessary, it will be assumed that $h(x) \varepsilon L_*^o$ has been extended over $(-\infty,a)$ by the definition $h(x) = h(a)$ for $x \varepsilon (-\infty,a)$.

An entire function $f(z)$ is said to be of generalized (α,β)-order $\rho_\infty(\alpha,\beta,f)$ and generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$ if

$$(1.3.3) \qquad \begin{matrix}\rho_\infty(\alpha,\beta,f) \\ \lambda_\infty(\alpha,\beta,f)\end{matrix} = \lim_{r \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\beta(\log r)}$$

where $\alpha(x) \varepsilon \Lambda_*$ and $\beta(x) \varepsilon L_*^o$. Taking $\alpha(x) = \log x$ and $\beta(x) = x$ in (1.3.3) we get the familiar order and lower order of an entire function $f(z)$, defined by (1.2.2) and (1.2.4). An entire function $f(z)$ for which $\rho_\infty(\alpha,\beta,f) = \lambda_\infty(\alpha,\beta,f)$ is said to be of generalized regular (α,β)-growth and $f(z)$ is said to be of generalized irregular (α,β)-growth if $\lambda_\infty(\alpha,\beta,f) < \rho_\infty(\alpha,\beta,f)$.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \tag{1.3.4}$$

$a_n \neq 0$ for every n, be an entire function. Then, following Šeremeta [93], it is seen that, if f(z) is of generalized (α,β)-order $\rho_\infty(\alpha,\beta,f)$, we have

$$\rho_\infty(\alpha,\beta,f) \geq \limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta(\frac{1}{\lambda_n} \log |a_n|^{-1})} \tag{1.3.5}$$

and, further, if $\alpha(x)$ and $\beta(x)$ satisfy

$$\frac{d\beta^{-1}(c\alpha(x))}{d \log x} = O(1) \quad \text{as } x \to \infty \tag{1.3.6}$$

for all c, $0 < c < \infty$, then

$$\rho_\infty(\alpha,\beta,f) = \limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta(\frac{1}{\lambda_n} \log |a_n|^{-1})} . \tag{1.3.7}$$

Šeremeta had obtained (1.3.5) and (1.3.7) for the case $\lambda_n = n$.

Coefficient equivalents of generalized lower (α,β)-order of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are due to Shah [102].However, we give here the coefficient equivalents of $\lambda_\infty(\alpha,\beta,f)$ for an entire function f(z) given by (1.3.4). In the present form, these results can be easily established by suitably adopting the techniques of Shah [102] and Juneja, Kapoor and Bajpai [45]. Thus, let f(z), given by (1.3.4), be an entire function of generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$. Assume that $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ is ultimately a nondecreasing function of n, $\alpha(x)$ and $\beta(x)$ satisfy

$$(1.3.8) \qquad \frac{d\beta^{-1}(\alpha(x))}{d \log x} = O(1) \text{ as } x \to \infty$$

and that there exists a function $\eta(x)$ such that $\eta(x) \to \infty$ as $x \to \infty$ and

$$(1.3.9) \qquad \frac{\beta(x\eta(x))}{\beta(e^x)} \to 0 \text{ as } x \to \infty,$$

then

$$(1.3.10) \qquad \lambda_\infty(\alpha,\beta,f) = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\beta(\frac{1}{\lambda_n} \log |a_n|^{-1})} .$$

Further, if $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9), then the following characterizations of generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$ of an entire function, given by (1.3.4), also hold in which the nondecreasing nature of the function $\Psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ is no more required

$$(1.3.11) \qquad \lambda_\infty(\alpha,\beta,f) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{\alpha(\lambda_{n_{k-1}})}{\beta(\frac{1}{\lambda_{n_k}} \log |a_{n_k}|^{-1})}\}$$

$$(1.3.12) \qquad \lambda_\infty(\alpha,\beta,f) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{\alpha(\lambda_{n_{k-1}})}{\beta(\frac{1}{\lambda_{n_k}-\lambda_{n_{k-1}}} \log |a_{n_{k-1}}/a_{n_k}|)}\}.$$

The maximum in (1.3.11) and (1.3.12) is taken over all increasing sequences $\{n_k\}$ of positive integers.

Some more results concerning generalized (α,β)-order and generalized lower (α,β)-order are found in [3,4].

1.4. Let $f(z)$ be an entire transcendental function, given by the power series (1.3.4). Since the series converges absolutely

for all finite z, $|a_n| r^{\lambda_n} \to 0$ as $n \to \infty$, for every finite r. Hence there is one term of the series whose absolute value is not less than that of any other term. The modulus of this term is denoted by $\mu(r,f)$ and is called the maximum term of f(z) for $|z| = r$. Thus,

$$\mu(r) \equiv \mu(r,f) = \max_{n \geq 0} \{|a_n| r^{\lambda_n}\}.$$

Let

$$\nu(r) \equiv \nu(r,f) = \max \{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\}.$$

Then $\nu(r)$ is called the central index of f(z) for $|z| = r$. The function $\nu(r)$ is a nondecreasing, integer valued, unbounded step function of r and has only left discontinuities. The elements $\{\lambda_{n_m}\}$ in the range set of $\nu(r)$ are called the principal indices of f(z) and the quantities $\tau(m) = \max \{r : \nu(r) = \lambda_{n_{m-1}}\}$ are called the jump points of the central index $\nu(r)$ of f(z).

By constructing Newton's polygon, the following relations involving the maximum term $\mu(r)$, the central index $\nu(r)$ and the maximum modulus M(r) of an entire function f(z) are established [126, pp. 28-32],

(1.4.1) $\log \mu(r) = \log \mu(r_o) + \int_{r_o}^{r} (\nu(t)/t)\, dt,\ 0 \leq r_o < r < \infty,$

(1.4.2) $\mu(r) \leq M(r) \leq \mu(r)\ (1 + 2\ \nu(r + \frac{r}{\nu(r)})).$

It has been shown that [126, p. 32], for an entire function f(z) of finite order $\rho_\infty(f)$, we have

(1.4.3) $\log M(r) \sim \log \mu(r)$ as $r \to \infty$

and that

(1.4.4) $$\rho_\infty(f) = \lim_{r \to \infty} \sup \frac{\log \log \mu(r)}{\log r} = \lim_{r \to \infty} \sup \frac{\log \nu(r)}{\log r} .$$

Whittaker [131] obtained an analogous result for the lower orde $\lambda_\infty(f)$. Thus, he proved that if $f(z)$, given by (1.3.4), is an entire function of order $\rho_\infty(f)$ and lower order $\lambda_\infty(f)$, then

(1.4.5) $$\lambda_\infty(f) = \lim_{r \to \infty} \inf \frac{\log \log \mu(r)}{\log r} = \lim_{r \to \infty} \inf \frac{\log \nu(r)}{\log r}$$

and

(1.4.6) $$\lambda_\infty(f) \leq \rho_\infty(f) \lim_{n \to \infty} \inf \frac{\log \lambda_n}{\log \lambda_{n+1}} .$$

It is easily seen that, in view of (1.4.3), the type $T_\infty(f)$ and lower type $t_\infty(f)$ of an entire function $f(z)$ of order $\rho_\infty(f)$, $0 < \rho_\infty(f) < \infty$, are given by

(1.4.7) $$\begin{matrix} T_\infty(f) \\ t_\infty(f) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \mu(r)}{r^{\rho_\infty(f)}} .$$

Shah [102] has extended the results (1.4.4) and (1.4.5) for generalized orders using a new technique. Thus, if $f(z)$ is an entire transcendental function of generalized (α,β)-order $\rho_\infty(\alpha,\beta,f)$ and generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$, then

(1.4.8) $$\begin{matrix} \rho_\infty(\alpha,\beta,f) \\ \lambda_\infty(\alpha,\beta,f) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log \mu(r))}{\beta(\log r)}$$

and, further, if (1.3.8) and (1.3.9) are satisfied, we have

$$(1.4.9) \qquad \begin{matrix} \rho_\infty(\alpha,\beta,f) \\ \lambda_\infty(\alpha,\beta,f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\nu(r))}{\beta(\log r)} .$$

Some other results involving $M(r)$, $\mu(r)$, $\nu(r)$, order, type etc. of an entire function have been found by Clunie [19], Clunie and Hayman [21,22], Grey and Shah [38], Polya and Szegö [78], Rahman [79,81], Shah [95,98,99,100], S.K.Singh [106], G.S. Srivastava [115], R.P. Srivastava [117], R.S.L.Srivastava [119] and others.

The growth of an entire function influences the distribution of its zeros and the results in this direction are found in [15], [31,32],[61], [76], [82] etc.

1.5. Let I denote the interval $[-1,1]$ and let $\mathcal{A}(I)$ be the class of all functions analytic in I. For $f(x) \in \mathcal{A}(I)$, the uniform norm is defined as .

$$(1.5.1) \qquad ||f||_{I,\infty} = \max_{x \in I} |f(x)|,$$

and the error $\Delta_{n,\infty}(f,I)$ in approximating the function $f(x)$ by polynomials of degree atmost n in uniform norm is defined as

$$(1.5.2) \qquad \Delta_{n,\infty}(f,I) = \inf_{g \in \mathbb{P}_n} ||f-g||_{I,\infty}$$

where $\mathbb{P}_n$ consists of all polynomials of degree atmost n.

The behaviour of the quantity $\Delta_{n,\infty}(f,I)$, as $n \to \infty$, was first investigated by Bernstein [12]. He showed that if $f(x) \in \mathcal{A}(I)$, then there exist ellipses ℓ_s with foci at -1 and $+1$ and sum of the semi-axes s such that $f(x)$ can be extended to

a function $f(z)$ analytic in their interiors. If $q = \sup\{s:$ is analytic in the interior of $\ell_s\}$, then ℓ_q is called the 'regularity ellipse' of $f(x) \varepsilon \mathcal{H}(I)$. If $q = \infty$, then ℓ_∞ con of the point ∞ only and $f(z)$ is entire. Further, he has show ([12, p. 118], [64, pp. 76-78], [71, pp. 90-94]) **that** $f(x) \varepsilon \mathcal{H}(I)$ can be extended to a function analytic in ℓ_q, if and only if,

$$(1.5.3) \qquad \limsup_{n \to \infty} (\Delta_{n,\infty}(f,I))^{1/n} = 1/q$$

so that $f(x)$ can be extended to an entire function, if and only if,

$$(1.5.4) \qquad \lim_{n \to \infty} (\Delta_{n,\infty}(f,I))^{1/n} = 0.$$

He has also shown that the rate of decrease of $\Delta_{n,\infty}(f,I)$ depe on the order and type of the entire function $f(z)$. Thus, ther exists a constant $\rho_\infty > 0$ such that

$$(1.5.5) \qquad \limsup_{n \to \infty} n(\Delta_{n,\infty}(f,I))^{\rho_\infty/n}$$

is finite, if and only if, $f(x) \varepsilon \mathcal{H}(I)$ can be extended to an entire function of order ρ_∞ and some finite type T_∞. Varga [128] strengthened it further when he showed that

$$(1.5.6) \qquad \rho_\infty \equiv \limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_{n,\infty}(f,I)}$$

is a nonnegative finite real number, if and only if, $f(x) \varepsilon \mathcal{H}($ can be extended to an entire function of order ρ_∞.

The above results of Bernstein and Varga were extended Reddy [83] for entire functions of fast growth. Juneja [43] Singh [105] obtained the analogous results for lower order. it was shown that if $f(x) \in \mathcal{H}(I)$ can be extended to an enti function of lower order $\lambda_\infty(f)$, then

$$(1.5.7) \qquad \lambda_\infty(f) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{n_k \log n_{k-1}}{-\log \Delta_{n_k,\infty}(f,I)}\}$$

where maximum is taken over all increasing sequences $\{n_k\}$ of positive integers.

Recently, Shah [102] has obtained characterizations of generalized (α,β)-order and generalized lower (α,β)-order of a entire function in terms of the approximation error $\Delta_{n,\infty}(f,I)$. Thus, if $f(x) \in \mathcal{H}(I)$ can be extended to an entire function ha generalized (α,β)-order $\rho_\infty(\alpha,\beta,f)$ and if (1.3.6) is satisfied, we have

$$(1.5.8) \qquad \rho_\infty(\alpha,\beta,f) = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(\frac{1}{n} \log (\Delta_{n,\infty}(f,I))^{-1})} .$$

Further, if $f(x) \in \mathcal{H}(I)$ is not a polynomial and can be extende to an entire function of generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f$ and if (1.3.8) and (1.3.9) are satisfied, we have

$$(1.5.9) \qquad \lambda_\infty(\alpha,\beta,f) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k} \log (\Delta_{n_k,\infty}(f,I))^{-1})}\},$$

(1.5.10) $\lambda_{\infty}(\alpha,\beta,f) =$

$$\max_{\{n_k\}}\{\liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k-n_{k-1}}\log \Delta_{n_{k-1},\infty}(f,I)/\Delta_{n_k,\infty}(f,I))}\}$$

where maximum in (1.5.9) and (1.5.10) is taken over all increas sequences $\{n_k\}$ of positive integers.

Some more results that depict the influence of the growth of entire functions on the rate of decrease of $\Delta_{n,\infty}(f,I)$ can be found in Kapoor [50], Reddy [84] etc.

1.6. Let $\mathcal{E}$ be a compact set in the complex plane and let $\zeta^{(n)} = \{\zeta_{no}, \zeta_{n1},\ldots,\zeta_{nn}\}$ be a system of n+1 points of the comp set $\mathcal{E}$. Set

$$V(\zeta^{(n)}) = \prod_{0\le j<k\le n} |\zeta_{nj} - \zeta_{nk}|,$$

$$\nabla^{(j)}(\zeta^{(n)}) = \prod_{\substack{k=0\\k\neq j}}^{n} |\zeta_{nj}-\zeta_{nk}|, \quad j = 0,1,\ldots,n.$$

A system of points

$$\eta^{(n)} = \{\eta_{no}, \eta_{n1},\ldots, \eta_{nn}\}$$

of $\mathcal{E}$ satisfying the relations

$$V(\eta^{(n)}) = \sup_{\zeta^{(n)}\subset\mathcal{E}} V(\zeta^{(n)}),$$

$$\nabla^{(0)}(\eta^{(n)}) \le \nabla^{(j)}(\eta^{(n)}), \quad j = 1,2,\ldots,n$$

is called the nth extremal system of $\mathcal{E}$ and the limit

(1.6.1) $$d(\mathcal{E}) = \lim_{n \to \infty} (V(\eta^{(n)}))^{2/(n(n+1))} = \lim_{n \to \infty} (V^{(o)}(\eta^{(n)}))^{1/n}$$

is called the transfinite diameter $d(\mathcal{E})$ ([36]) of $\mathcal{E}$.

Results of the previous section have been extended to more general sets in the complex plane. Thus, let $\mathcal{E}$ be a compact s whose transfinite diameter $d(\mathcal{E}) > 0$ and let $\mathcal{H}(\mathcal{E})$ be the clas of all functions $f(z)$ continuous on $\mathcal{E}$. For $f(z) \in \mathcal{H}(\mathcal{E})$, set

(1.6.2) $$||f||_{\mathcal{E},\infty} = \max_{z \in \mathcal{E}} |f(z)|.$$

Then, $||\cdot||_{\mathcal{E},\infty}$ is called uniform norm on $\mathcal{H}(\mathcal{E})$. For $f(z) \in \mathcal{H}($ let the error $\Delta_{n,\infty}(f)$ in approximating the function $f(z)$ by polynomials of degree atmost n be defined as

(1.6.3) $$\Delta_{n,\infty}(f) \equiv \Delta_{n,\infty}(f,\mathcal{E}) = \inf_{g \in \mathbb{P}_n} ||f-g||_{\mathcal{E},\infty}.$$

where $\mathbb{P}_n$ consists of all polynomials of degree atmost n.

Winiarski [132] has shown that $f(z) \in \mathcal{H}(\mathcal{E})$ can be extended to an entire function of order $\rho_\infty(f)$ $(0 < \rho_\infty(f) < \infty)$ and type $T_\infty(f)$ $(0 < T_\infty(f) < \infty)$, if and only if,

(1.6.4) $$W_\infty(f) \equiv \limsup_{n \to \infty} n(\Delta_{n,\infty}(f))^{\rho_\infty(f)/n}$$

satisfies $0 < W_\infty(f) < \infty$ and, further $W_\infty(f) = e\, \rho_\infty(f) T_\infty(f) (d(\mathcal{E}))^{\rho_\infty}$ holds, where $d(\mathcal{E})$ is the transfinite diameter of $\mathcal{E}$. He has also shown that $f(z) \in \mathcal{H}(\mathcal{E})$ can be extended to an entire function of finite order $\rho_\infty(f)$, if and only if,

(1.6.5) $$\rho_\infty(f) = \limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_{n,\infty}(f)} < \infty.$$

Juneja and Rizvi [47] have obtained characterizations of lower order of an entire function in terms of the approximati error $\Delta_{n,\infty}(f,\mathcal{E})$. Characterizations of generalized (α,β)-or and generalized lower (α,β)-order of an entire function, in terms of the error $\Delta_{n,\infty}(f,\mathcal{E})$, have been found by Bajpai and Shah [6]. Some more results in this direction are due to Batyrev [10], Rice [86], Rizvi [87] and others.

1.7. Let D denote a domain bounded by a closed Jordan curve. For $1 \leq \delta < \infty$, let $L^{\delta}(D)$ be the class of all functions $f(z)$ analytic in D and satisfying

$$||f||_{D,\delta} = \Big(\iint_D |f(z)|^{\delta}\, dx\, dy\Big)^{1/\delta} < \infty.$$

Then $||\cdot||_{D,\delta}$, $1 \leq \delta < \infty$, is L^{δ}-norm on $L^{\delta}(D)$. The error $\Delta_{n,\delta}(f,D)$ in approximating the function $f(z) \in L^{\delta}(D)$ by polynomials of degree atmost n in L^{δ}-norm is defined as

$$\Delta_{n,\delta}(f,D) = \inf_{g \in \mathbb{P}_n} ||f-g||_{D,\delta}\ ,\ 1 \leq \delta < \infty,$$

where $\mathbb{P}_n$ consists of all polynomials of degree atmost n.

Giroux [35], has shown that if $2 \leq \delta < \infty$, then $f(z) \in L^{\delta}($ can be extended to an entire function, if and only if,

(1.7.1) $$\lim_{n \to \infty} (\Delta_{n,\delta}(f,D))^{1/n} = 0.$$

Further, $f(z) \in L^{\delta}(D)$, $2 \leq \delta < \infty$, can be extended to an entire function of finite order $\rho_{\infty}(f)$, if and only if,

(1.7.2) $$\rho_{\infty}(f) = \limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_{n,\delta}(f,D)} < \infty$$

Further, f(z) is of order $\rho_\infty(f)$, $0 < \rho_\infty(f) < \infty$, and of type $T_\infty($
$0 < T_\infty(f) < \infty$, if and only if,

$$(1.7.3) \quad e\,\rho_\infty(f)\,T_\infty(f)\,(d(\bar{D}))^{\rho_\infty(f)} = \limsup_{n \to \infty} n(\Delta_{n,\delta}(f,D))^{\rho_\infty(f)/n}$$

where $d(\bar{D})$ is the transfinite diameter of $\bar{D}$, the closure of D.

For $\delta = 2$, the results (1.7.1) to (1.7.3) were obtained earlier by Rizvi [87]. For $\delta = 2$ and $D = \{z: |z| < 1\}$ these results were obtained by Reddy [85]. For the case $D = \{z: |z| < 1$ Ibragimov and Šihaliev [39] have extended the results of Reddy and have obtained the relations (1.7.1) to (1.7.3) for $1 \leq \delta < \infty$ Taking D to be a Jordan domain bounded by finite number of analytic Jordan arcs meeting in corners of exterior openings les than or equal to $s\pi < 2\pi$, Rizvi [87] obtained the relations (1.7. to (1.7.3) for the case $1 \leq \delta < \infty$. For $\delta = 2$ and $D = \{z: |z| < 1\}$ Bajpai and Shah [6] have obtained characterizations of generaliz (α,β)-order and generalized lower (α,β)-order of an entire functi in terms of the error $\Delta_{n,2}(f,D)$. Some more results in this direction can be found in Rizvi [87].

1.8. The results reviewed in the previous sections clearly demonstrate the relationship that exists between the growth of an entire function and its degree of approximation by polynomials on certain sets in the complex plane. Now, we give some results concerning the growth and the degree of approximation of functions analytic in a finite region that are relevant to the present study.

In this section we first have some results concerning the growth measurement of analytic functions.

Recently, considerable interest has been shown by differen workers (e.g., [1], [8], [16,17], [49-52], [53-55], [56], [62], [110] etc.) to study the growth of functions that are not entir but are analytic in a finite disc $D_R = \{z : |z| < R\}$, $0 < R < \infty$ Analogous to the case of entire functions the growth of these functions has been studied in terms of the maximum modulus $M(r,f)$ of $f(z)$ on $|z| = r < R$. Thus, a function $f(z)$, analytic in D_R, $0 < R < \infty$, is said to be of order $\rho(f)$ and lower order $\lambda(f)$ $(0 \leq \lambda(f) \leq \rho(f) \leq \infty)$ if

$$(1.8.1) \qquad \begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^+ \log^+ M(r,f)}{\log (R/(R-r))}$$

where $\log^+ x = \max (0, \log x)$. As usual, a function $f(z)$, analytic in D_R, $0 < R < \infty$, is said to be of regular growth if $\rho(f) = \lambda(f)$ and $f(z)$ is said to be of irregular growth if $\lambda(f) < \rho(f)$.

A function $f(z)$, analytic in D_R, $0 < R < \infty$, is said to be of fast growth if $\rho(f) = \infty$ and of slow growth if $\rho(f) = 0$.

For a more precise specification of the growth of a functior $f(z)$, analytic in D_R, $0 < R < \infty$, and having order $\rho(f)$, $0 < \rho(f) < \infty$, the concepts of type $T(f)$ and lower type $t(f)$ of $f(z)$ are introduced in ([8], [50]) as follows

$$(1.8.2)\qquad \begin{matrix} T(f) \\ t(f) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^+ M(r,f)}{(R/(R-r))^{\rho(f)}} .$$

The concepts of the maximum term and central index, for a function $f(z)$, analytic in D_R, $0 < R < \infty$, can be introduced i in the same manner as done for entire functions. Thus, if

$$(1.8.3)\qquad f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n},$$

$a_n \neq 0$ for all n, is a function analytic in D_R, $0 < R < \infty$, then its maximum term $\mu(r)$ and the central index $\nu(r)$, for $0 < r < R$, are defined as

$$\mu(r) \equiv \mu(r,f) = \max_{n \geq 0} \{|a_n| r^{\lambda_n}\},$$

$$\nu(r) \equiv \nu(r,f) = \max \{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\}.$$

The elements $\{\lambda_{n_m}\}$ in the range set of $\nu(r)$ are called the principal indices of $f(z)$ and the quantities $\tau(m) = \max \{r: \nu(r)$ $\lambda_{n_{m-1}}\}$ are called the jump points of the central index $\nu(r)$ of $f(z)$.

If $\mu(r)$ and $\nu(r)$ are unbounded functions of r, then the following relations involving the maximum term $\mu(r)$, central , index $\nu(r)$ and the maximum modulus $M(r)$ of a function $f(z)$, analytic in D_R, $0 < R < \infty$, are easily established on the lines used in [127, pp. 195-201] for establishing these relations for the case $R = 1$

$$(1.8.4)\qquad \log \mu(r) = \log \mu(r_o) + \int_{r_o}^{r} (\nu(t)/t)\, dt,$$

(1.8.5) $\quad \mu(r) \leq M(r) \leq \mu(r) \; \{1+2\nu(r + \frac{R-r}{\nu(r)})\} \; \frac{R}{R-r}, \; 0 < r_o < r <$

Sons [110, 112] obtained the order and lower order of a function f(z), analytic in D_1, in terms of its maximum term and the central index. Thus, she showed that; if the function f(z), given by (1.8.3) and analytic in D_1, has order $\rho(f)$ $(0 < \rho(f) < \infty)$ and lower order $\lambda(f)$ $(0 \leq \lambda(f) < \infty)$, then

$$(1.8.6) \qquad \begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log \mu(r)}{-\log (1-r)}$$

and

$$(1.8.7) \qquad 1 + \rho(f) = \limsup_{r \to 1} \frac{\log \nu(r)}{-\log (1-r)} .$$

Further [110, 112]

$$(1.8.8) \qquad \lambda(f) \leq \liminf_{r \to 1} \frac{\log \nu(r)}{-\log (1-r)} \leq 1+\lambda(f)$$

and

$$(1.8.9) \qquad \lambda(f) \leq \rho(f) \liminf_{n \to \infty} \frac{\log \lambda_n}{\log \lambda_{n+1}} .$$

A result analogous to (1.8.7), in place of (1.8.8), does not hold, in general, for the lower order $\lambda(f)$ of analytic functions. For, consider the function $f_o(z) = \sum_{k=0}^{\infty} \exp(\sqrt{n_k}) \; z^{n_k}$, where $\{n_k\}$ is an increasing sequence of natural numbers such that $n_{k+1} = n_k^2$. It is easily seen that $f_o(z)$ is analytic in D_1, $\rho(f_o) = 1$, $\lambda(f_o) = 1/2$ and $\liminf_{r \to 1} (\log \nu(r))/(-\log (1-r)) = 1$. This example is due to Paul V. Reichelderfer.

Some other results connecting $\mu(r)$, $\nu(r)$, order, lower orde. type and lower type of a function, analytic in D_R, $0 < R < \infty$, are to be found in Bajpai and Tanne [7], Bogda and Shankar [16], Kapoor [52], Kővari [59], Srivastava and Juneja [116] etc.

Some results depicting the interrelations between the growth of a function, analytic in a finite disc, and the distribution of its zeros could be found in Linden [62], Sons [109,111,113,114], Tsuji [124,125], etc.

Coefficient characterization of the order $\rho(f)$ of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in D_1, was obtained separately by Beuermann [13] and MacLane [65]. Kapoor [50,51] and Kapoor and Juneja [56] have characterized lower order of a function, analytic in D_1, in terms of its Taylor coefficients. Coefficient characterizations of type and lower type of a function, analytic in D_R, $0 < R < \infty$, were obtained, by Bajpai, Tanne and Whittier [8] and Kapoor [49,50]. However, all these results are included in the work of Kapoor and Gopal [53,55], who have given a new scheme to study the growth of analytic functions of infinite order. We discuss this scheme in the next section.

1.9. For a function $f(z)$ analytic in D_R, $0 < R < \infty$, set

$$\rho(q,f) = \limsup_{r \to R} \frac{\log_q M(r,f)}{\log (R/(R-r))}, \quad q = 2,3,\ldots . \tag{1.9.1}$$

Then $f(z)$ is said to be of index q if $\rho(q,f) < \infty$ and $\rho(q-1,f) = \infty$, $q = 2,3,\ldots$. If q is the index of $f(z)$, then $\rho(q,f)$ is called

the q-order of f(z) ([53], [37]) and the lower q-order of f(z) is defined as

$$(1.9.2) \qquad \lambda(q,f) = \liminf_{r \to R} \frac{\log_q M(r,f)}{\log (R/(R-r))} .$$

Further, if f(z), analytic in D_R, $0 < R < \infty$, is of index q with q-order $\rho(q,f) > 0$, then the q-type and lower q-type of f(z) are defined in [53,55] and [37] as

$$(1.9.3) \qquad \begin{matrix} T(q,f) \\ t(q,f) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log_{q-1} M(r,f)}{(R/(R-r))^{\rho(q,f)}} .$$

For q = 2, $\rho(q,f)$ and $\lambda(q,f)$ give the order and lower order, defined by (1.8.1), while $T(q,f)$ and $t(q,f)$ give the type and lower type, defined by (1.8.2), of a function f(z) analytic in D_R $0 < R < \infty$.

Coefficient characterizations of q-order, lower q-order, q-type and lower q-type have been found in Kapoor and Gopal [53, 55] and [37]. Thus, the q-order $\rho(q,f)$ of a function f(z), given by (1.8.3) and analytic in D_R, $0 < R < \infty$, is given by

$$(1.9.4) \qquad \rho(q,f) + X(q) = \limsup_{n \to \infty} \frac{\log_{q-1} \lambda_n}{\log \lambda_n - \log^+ \log^+ |a_n| R^{\lambda_n}}$$

where

$$(1.9.5) \qquad X(q) = \begin{cases} 1 & \text{if } q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

For q = 2, R = 1 and $\lambda_n = n$, (1.9.4) was obtained by Beuermann [13] and MacLane [65]. For $q \geq 3$, lower q-order $\lambda(q,f)$ of f(z), given by (1.8.3) and analytic in D_R, $0 < R < \infty$, is given by

$$(1.9.6)\quad \lambda(q,f)+X(q) = \max_{\{n_k\}} \{\liminf_{k\to\infty} \frac{\log_{q-1}\lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+\log^+|a_{n_k}|R^{\lambda_{n_k}}}\}$$

where $X(q)$ is given by (1.9.5) and maximum in (1.9.6) is taken over all increasing sequences $\{n_k\}$ of positive integers; for the case $q = 2$, (1.9.6) continues to hold provided the principal indices $\{\lambda_{n_m}\}$ of the function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ satisfy $\log \lambda_{n_{m-1}} \sim \log \lambda_{n_m}$ as $m \to \infty$. For the case $q = 2$, some other coefficient equivalents of lower q-order, analogous to (1.9.6), have been found by Kapoor [51] and Kapoor and Juneja [56].

It has also been shown that([53,55], [37]), if $f(z)$ given by (1.8.3) and analytic in D_R, $0 < R < \infty$, is of q-order $\rho(q,f)$ $(0 < \rho(q,f) < \infty)$, q-type $T(q,f)$ and lower q-type $t(q,f)$, then

$$(1.9.7)\quad T(q,f)\ B(q) = \limsup_{n\to\infty} (\log_{q-2}\lambda_n)\left(\frac{\log^+ |a_n|\ R^{\lambda_n}}{\lambda_n}\right)^{\rho(q,f)+X(q)}$$

where

$$(1.9.8)\quad B(q) = \begin{cases} (\rho(f)+1)^{\rho(f)+1}/\rho(f)^{\rho(f)} & \text{if } q = 2 \\ 1 & \text{if } q = 3,4,\ldots \end{cases}$$

and $X(q)$ is given by (1.9.5); further, if $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ is ultimately a nondecreasing function of n and $\log_{q-2}\lambda_n \sim \log_{q-2}\lambda_{n+1}$ as $n \to \infty$, then

$$(1.9.9)\quad t(q,f)\ B(q) = \liminf_{n\to\infty} (\log_{q-2}\lambda_{n-1})\left(\frac{\log^+ |a_n|R^{\lambda_n}}{\lambda_n}\right)^{\rho(q,f)+X(q)},$$

where $X(q)$ and $B(q)$ are given by (1.9.5) and (1.9.8), respectively. It has also been shown ([53,55]) that if the principal indices $\{\lambda_{n_m}\}$ of the function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ satisfy $\log_{q-2} \lambda_{n_m} \sim \log_{q-2} \lambda_{n_{m+1}}$ as $m \to \infty$, we have

$$(1.9.10) \qquad t(q,f)\, B(q) = \max_{\{n_k\}} \Big\{ \liminf_{k \to \infty} (\log_{q-2} \lambda_{n_{k-1}}) \times \Big(\frac{\log^+ |a_{n_k}| R^{\lambda_{n_k}}}{\lambda_{n_k}} \Big)^{\rho(q,f)+X(q)} \Big\}$$

where $X(q)$ and $B(q)$ are given by (1.9.5) and (1.9.8), respectively and maximum in(1.9.10) is taken over all increasing sequences $\{n_k\}$ of positive integers.

For $q = 2$, (1.9.7) and (1.9.9) were obtained by Kapoor [49,50] and Bajpai, Tanne and Whittier [8], while (1.9.10) was obtained by Kapoor [50]. Some other results concerning q-type and lower q-type are to found in [49,50], [37] , etc.

It is easily seen that for functions, analytic in D_R, $0 < R < \infty$, and having slow rate of growth the concept of q-order does not give any specific information about their growth. Thus, to study precisely the growth of analytic functions having zero order the concepts of logarithmic order and lower logarithmic order have been introduced and investigated in [54] and [37]. A function $f(z)$, analytic in D_R, $0 < R < \infty$, is said to be of logarithmic order $\rho^*(f)$ and lower logarithmic order $\lambda^*(f)$, if

$$(1.9.11)\qquad \begin{matrix}\rho^*(f)\\ \lambda^*(f)\end{matrix} = \lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^+\log^+ M(r,f)}{\log\log (R/(R-r))} .$$

Kapoor and Gopal [54] have shown that, if $f(z)$, given by (1.8.3) and analytic in D_R, $0 < R < \infty$, is of logarithmic order $\rho^*(f)$, then

$$(1.9.12)\qquad U \le \rho^*(f) \le \max(1,U)$$

where

$$U = \limsup_{n\to\infty} \frac{\log^+\log^+ |a_n|\, R^{\lambda_n}}{\log\log \lambda_n} .$$

Further, if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, is of lower logarithmic order $\lambda^*(f)$ with $\lambda^*(f) \ge 1$ and the principal indices $\{\lambda_{n_m}\}$ of the function $f(z)$ satisfy $\log\log \lambda_{n_m} \sim \log\log \lambda_{n_{m+1}}$ as $m \to \infty$, then it has been shown that

$$(1.9.13)\qquad \lambda^*(f) = \max(1,U^*)$$

where

$$U^* = \max_{\{n_k\}} \{\liminf_{k\to\infty} \frac{\log^+\log^+ |a_{n_k}|\, R^{\lambda_{n_k}}}{\log\log \lambda_{n_{k+1}}}\}$$

and the maximum has been taken over all increasing sequences $\{n_k\}$ of positive integers.

To compare the rates of growth of analytic functions having the same logarithmic order the concepts of logarithmic type and lower logarithmic type have been introduced and investigated in [54] and [37]. Thus, if the logarithmic order $\rho^*(f)$ of a

function f(z), analytic in D_R, $0 < R < \infty$, satisfies $1 < \rho^*(f)$
then the logarithmic type $T^*(f)$ and lower logarithmic type t^*
of f(z) are defined as

$$(1.9.14)\qquad \begin{matrix} T^*(f) \\ t^*(f) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r,f)}{(\log (R/(R-r)))^{\rho^*(f)}} .$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, is of logarit[hmic]
order $\rho^*(f)$ with $1 < \rho^*(f) < \infty$, logarithmic type $T^*(f)$ and low[er]
logarithmic type $t^*(f)$, then [54] , we have

$$(1.9.15)\qquad T^*(f) = \limsup_{n \to \infty} \frac{\log^+ |a_n| R^{\lambda_n}}{(\log \lambda_n)^{\rho^*(f)}}$$

and if, we further assume that the principal indices $\{\lambda_{n_m}\}$ of
the function f(z) satisfy $\log \lambda_{n_m} \sim \log \lambda_{n_{m+1}}$ as $m \to \infty$, we hav[e]

$$(1.9.16)\qquad t^*(f) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{\log^+ |a_{n_k}| R^{\lambda_{n_k}}}{(\log \lambda_{n_{k+1}})^{\rho^*(f)}}\}$$

where maximum in (1.9.16) is taken over all increasing sequence[s]
$\{n_k\}$ of positive integers.

Some more results concerning logarithmic order, lower logarithmic order, logarithmic type and lower logarithmic type are found in [54] and [37].

Šeremeta [92] has studied the growth of a function f(z), analytic in the unit disc D_1, by using more general functions

than the iterates of logarithm for comparing the growth of $\log M(r,f)$ with that of $1/(1-r)$. Thus, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D_1 and let the class of functions Λ_* be as defined in Section 1.3. In the statements of the following results, wherever necessary, we shall assume that $h(x) \in \Lambda_*$ has been extended over $(-\infty, a)$ by the definition $h(x) = h(a)$ for $x \in (-\infty, a)$.

For $\alpha(x)$, $\beta(x) \in \Lambda_*$, let $F(x,c) = \beta^{-1}(c\alpha(x))$. If, for $0 < c < \sigma$, where $\sigma = \infty$ if $\alpha \not\equiv \beta$ and $\sigma = 1$ if $\alpha \equiv \beta$,

(1.9.17) $$\alpha(x/F(x,c)) \sim \alpha(x) \text{ as } x \to \infty$$

and

(1.9.18) $$\limsup_{x \to \infty} \frac{d \log F(x,c)}{d \log x} < 1$$

then, for $\alpha \not\equiv \beta$, we have

(1.9.19) $$\limsup_{r \to 1} \frac{\alpha(\log M(r,f))}{\beta(1/(1-r))} = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(n/\log^+ |a_n|)}$$

and, further, (1.9.19) continues to hold for the case $\alpha \equiv \beta$ if the limit superior on the left hand side of (1.9.19) is greater than one.

Šeremeta [92] has further shown that under suitable conditions on $\alpha(x) \in \Lambda_*$, for the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in D_1, we have

(1.9.20) $$\limsup_{r \to 1} \frac{\alpha(\log M(r,f))}{\alpha(\log(1/(1-r)))} = \limsup_{n \to \infty} \frac{\alpha(\log^+ |a_n|)}{\alpha(\log n)}$$

if the limit superior on the left hand side of (1.9.20) is greater than one.

A result analogous to (1.9.19), under various restrictions on $\alpha(x)$ and $\beta(x)$, has also been found in [1].

1.10. Some of the results of Sections 1.8 and 1.9 have recent found application in studying the growth of functions, analytic in a finite region, in terms of their degree of approximation. In this section we discuss some such results that are relevant to the present study.

Let $\mathcal{E}_o$ be a compact set, containing more than one point, such that the complement $\mathcal{E}_o^*$ of $\mathcal{E}_o$ is a simply connected domain. Let $\mathcal{H}(\mathcal{E}_o)$ be the linear space of all functions holomorphic on $\mathcal{E}_o$. For $f(z) \in \mathcal{H}(\mathcal{E}_o)$, let the error $\Delta_{n,\infty}(f)$ in approximatin the function $f(z)$ by polynomials of degree atmost n be defined as

$$\Delta_{n,\infty}(f) \equiv \Delta_{n,\infty}(f, \mathcal{E}_o) = \inf_{g \in \mathbb{P}_n} \|f-g\|_{\mathcal{E}_o,\infty},$$

where $\|f\|_{\mathcal{E}_o,\infty} = \sup_{z \in \mathcal{E}_o} |f(z)|$ is the uniform norm and $\mathbb{P}_n$ consists of all polynomials of degree atmost n.

Let $\mathcal{E}_{o,r} = \{z : |\phi(z)| = r\}$, $r > 1$, where the univalent function $w = \phi(z)$ maps $\mathcal{E}_o^*$ onto $|w| > 1$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Then $\mathcal{E}_{o,r}$ is an analytic Jordan curve. Let $\mathcal{J}(\mathcal{E}_{o,r})$ be the domain bounded by the Jordan curve $\mathcal{E}_{o,r}$. Then, $\mathcal{E}_o \subset \mathcal{J}(\mathcal{E}_{o,r})$ for each r, $1 < r < \infty$, and $\mathcal{E}_{o,r} \subset \mathcal{J}(\mathcal{E}_{o,r'})$ for every $r' > r$. Since, through an arbitrary point $z_o \notin \mathcal{E}_o$, there passes one and only one curve $\mathcal{E}_{o,r}$, $1 < r < \infty$, it follows

that for each $f(z) \in \mathcal{H}(\mathcal{E}_o)$ there exists a unique $R \equiv R(f)$ $(1 < R \leq \infty)$ such that $f(z)$ can be extended analytically to $\mathcal{I}(\mathcal{E}_{o,r})$ for every $r \leq R$ but for no $r > R$. Then, $\mathcal{I}(\mathcal{E}_{o,R})$ is called the 'regularity domain' for $f(z)$. Let, $\mathcal{H}(\mathcal{E}_o, R)$ be the class of all those functions $f(z) \in \mathcal{H}(\mathcal{E}_o)$ which have 'regulari domain' $\mathcal{I}(\mathcal{E}_{o,R})$.

Let

$$\bar{M}(r,f) = \max_{z \in \mathcal{E}_{o,r}} |f(z)|.$$

Then the order $\rho_o(f)$ of $f(z) \in \mathcal{H}(\mathcal{E}_o, R)$, $1 < R < \infty$, is defined as ([87], [88])

$$\rho_o(f) = \limsup_{r \to R} \frac{\log^+ \log^+ \bar{M}(r,f)}{\log (R/(R-r))}$$

and, further, if $0 < \rho_o(f) < \infty$, the type $T_o(f)$ of $f(z)$ is defined as

$$T_o(f) = \limsup_{r \to R} \frac{\log^+ \bar{M}(r,f)}{(R/(R-r))^{\rho_o(f)}} .$$

It has been shown ([87], [88]) that a function $f(z) \in \mathcal{H}(\mathcal{E}_o$ belongs to $\mathcal{H}(\mathcal{E}_o, R)$, $1 < R \leq \infty$, if and only if,

$$\limsup_{n \to \infty} (\Delta_{n,\infty}(f))^{1/n} = 1/R.$$

This generalizes a corresponding result of Bernstein, given by (1.5.3), obtained for the case when $\mathcal{E}_o$ is taken to be the closed interval $[-1,1]$.

Characterizations of the order $\rho_o(f)$ and type $T_o(f)$ of $f(z) \in \mathcal{H}(\mathcal{E}_o, R)$, $1 < R < \infty$, in terms of the approximation error

$\Delta_{n,\infty}(f)$ have been obtained in [87] and [88]. Thus, the order $\rho_o(f)$ of $f(z) \in \mathcal{H}(\mathcal{E}_o, R)$, $1 < R < \infty$, is given by

$$\rho_o(f) = \limsup_{n \to \infty} \frac{\log^+ \log^+ \Delta_{n,\infty}(f)\ R^n}{\log n - \log^+ \log^+ \Delta_{n,\infty}(f)\ R^n},$$

and if $0 < \rho_o(f) < \infty$, the type $T_o(f)$ of $f(z)$ is given by

$$\frac{(\rho_o(f)+1)^{\rho_o(f)+1}}{\rho_o(f)^{\rho_o(f)}}\, T_o(f) = \limsup_{n \to \infty} \frac{(\log^+ \Delta_{n,\infty}(f)\ R^n)^{\rho_o(f)+1}}{n^{\rho_o(f)}}.$$

Some more results concerning the approximation of functions, analytic a finite region, have been found in [87] and [75].

1.11. The previous sections dealt with some of the results concerning growth and approximation of analytic and entire functions of a single complex variable. We now give some results concerning the growth and approximation of entire solutions of certain partial differential equations that are relevant to the present study.

Harmonic functions in R^n, $n \geq 2$, are the solutions of the n-dimensional Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \cdots + \frac{\partial^2 H}{\partial x_n^2} = 0. \tag{1.11.1}$$

A solution H of (1.11.1), which has continuous second derivatives in some neighbourhood N_o of the origin, is said to be a harmonic function regular about origin. A harmonic function H, regular about origin, can be expanded as ([23, p. 269], [77,p.45])

$$(1.11.2) \qquad H(x_1,x_2,\dots,x_n) = \sum_{k=0}^{\infty} H_k(x_1,x_2,\dots,x_n)$$

where $H_k(x_1,x_2,\dots,x_n)$ is a homogeneous polynomial of degree k in $x_1,\dots,x_n$ satisfying (1.11.1).

A harmonic function H, given by (1.11.2), is said to be entire if the series on the right hand side of (1.11.2) converg uniformly on compact subsets of R^n. For an entire harmonic function H, set

$$M(r,H) = \max_{x_1^2+\dots+x_n^2=r^2} |H(x_1,x_2,\dots,x_n)|.$$

The order $\rho_\infty(H)$ of an entire harmonic function H is defined as ([30])

$$(1.11.3) \qquad \rho_\infty(H) = \limsup_{r\to\infty} \frac{\log\log M(r,H)}{\log r}$$

and, further, if $0 < \rho_\infty(H) < \infty$, the type $T_\infty(H)$ of H is defined a

$$(1.11.4) \qquad T_\infty(H) = \limsup_{r\to\infty} \frac{\log M(r,H)}{r^{\rho_\infty(H)}} .$$

Analogous to the lower type of an entire function we define the lower type $t_\infty(H)$ of an entire harmonic function having order $\rho_\infty(H)$, $0 < \rho_\infty(H) < \infty$, as

$$(1.11.5) \qquad t_\infty(H) = \liminf_{r\to\infty} \frac{\log M(r,H)}{r^{\rho_\infty(H)}} .$$

For an entire harmonic function H, Fryant [27] and Fugard [30] have obtained expressions for order $\rho_\infty(H)$ and type $T_\infty(H)$ in terms of its derivatives at the origin.

Fryant has studied the harmonic function in R^3. Thus, for an entire harmonic function H in R^3 he considered the following expansion in spherical harmonics [34, p. 47]

(1.11.6) $$H(x_1,x_2,x_3) \equiv H(r,\theta,\phi) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (a_{km}^{(1)} \cos m\phi + a_{km}^{(2)} \sin m\phi) r^k P_k^m (\cos\theta)$$

where $x_1 = r \cos\theta$, $x_2 = r \sin\theta \cos\phi$, $x_3 = r \sin\theta \sin\phi$ and $P_k^m(t)$ are associated Legendre's functions of first kind, kth degree and order m and are given by

$$P_k^m(t) = \frac{(1-t^2)^{m/2}}{2^k\, k!} \frac{d^{k+m}(t^2-1)^k}{dt^{k+m}}.$$

It is known ([11, pp. 41-43]) that Bergman's $\mathcal{B}_3$ integral operator maps a function $h(u,\zeta)$ of two complex variables onto a given harmonic function $H(x_1,x_2,x_3)$, regular about origin in R^3, and is given by

(1.11.7) $$H(x_1,x_2,x_3) = \mathcal{B}_3(h(u,\zeta)) = \frac{1}{2\pi i} \int_{|\zeta|=1} h(u,\zeta)\, \zeta^{-1}\, d\zeta$$

where $u = x+iy\,(\zeta+\zeta^{-1})/2 + z(\zeta-\zeta^{-1})/2$ and h is analytic in u and continuous in ζ for all ζ on the unit circle.

Using Bergman's $\mathcal{B}_3$ integral operator Fryant [27] has shown that if an entire harmonic function H in R^3, given by (1.11.6), is of order $\rho_\infty(H)$, then

$$(1.11.8)\qquad \rho_\infty(H) = \limsup_{k\to\infty}\{\max_{m,i} \frac{k \log k}{\log|a_{km}^{(i)}|^{-1}+\frac{1}{2}\log(k-m)!/(k+m)!}\}$$

and, further, if $0 < \rho_\infty(H) < \infty$, then the type $T_\infty(H)$ of H satisfies

$$(1.11.9)\qquad L \le T_\infty(H) \le 2^{\rho_\infty(H)} L$$

where

$$L = \frac{1}{e\rho_\infty(H)} \limsup_{k\to\infty} k \max_{m,i} \{(\frac{(k+m)!}{(k-m)!})^{1/2} |a_{km}^{(i)}|\}^{\rho_\infty(H)/k}.$$

For each n-tuple $\tilde{a} = (a_1,a_2,\ldots,a_n)$ of nonnegative integers let

$$|\tilde{a}| = a_1+a_2+\ldots+a_n,\ \tilde{a}! = a_1!a_2!\ldots a_n!,\ D^{\tilde{a}} = \frac{\partial^{|\tilde{a}|}}{\partial x_1^{a_1}\ldots\partial x_n^{a_n}}.$$

Then, for an entire harmonic function H in R^n, $n \ge 2$, Fugard [30] has defined $|\nabla_m H|$, $m = 1,2,\ldots$, the norm of the mth gradient of H as

$$(1.11.10)\qquad |\nabla_m H| = \{m! \sum_{|\tilde{a}|=m} (D^{\tilde{a}}H)^2 (\tilde{a}!)^{-1}\}^{1/2}.$$

He has shown that the order $\rho_\infty(H)$ of an entire harmonic function H in R^n, $n \ge 2$, is given by

$$(1.11.11)\qquad \rho_\infty(H) = \limsup_{m\to\infty} \frac{m \log m}{\log (m!/|\nabla_m H(0)|)}$$

and, further if $0 < \rho_\infty(H) < \infty$, the type $T_\infty(H)$ of H is given by

$$T_\infty(H) = 2^{-\rho_\infty(H)/2} (e\rho_\infty(H))^{-1} \limsup_{m\to\infty} m(\frac{|\nabla_m H(0)|}{m!})^{\rho_\infty(H)/m}.$$

In [18], Calderon and Zygmund have investigated a norm of mth gradient, analogous to (1.11.10).

Some more results concerning the growth of entire harmonic functions can be found in [2], [63] etc.

1.12. The solutions of the n-dimensional Laplace equation (1.11.1) that depend on the variables

$$x = x_1, \; y = (x_2^2 + \dots + x_n^2)^{1/2}$$

are called axisymmetric potentials and they satisfy the partial differential equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{n-2}{y}\frac{\partial G}{\partial y} = 0.$$

The solutions of the partial differential equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{2u}{y}\frac{\partial G}{\partial y} = 0, \; u > 0, \tag{1.12.1}$$

were first investigated by Weinstein [130] and called generalize axisymmetric potentials (GASP's). A polynomial of degree n in x and y that satisfies (1.12.1) is said to be a GASP polynomial of degree n. Let π_n^* denote the class of all GASP polynomials of degree atmost n.

A solution G of (1.12.1), which has continuous second derivatives in some neighbourhood N_o of the origin and satisfies $\partial G(x,0)/\partial y = 0$ on the intersection of N_o with x axis, is said to be a GASP regular about origin. It is known [34, p.173] that a GASP G, regular about the origin, has the following ultraspherical harmonic expansion

(1.12.2) $$G(x,y) \equiv G(r,\theta) = \sum_{n=0}^{\infty} b_n\, r^n\, C_n^u(\cos\theta),$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $C_n^u(x)$ are Gegenbauer polynomials of degree n, given by ([122, p. 97])

$$C_n^u(x) = \frac{2^{1-2u}}{\Gamma(u)}\,\frac{\Gamma(n+2u)}{n!}\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{\Gamma(k+1/2)}{\Gamma(u+k+1/2)}\, x^{n-2k}(1-x^2)^k$$

where $[n/2]$ denotes the integral part of $n/2$.

It is known that Gilbert's A_u-operator [34, pp. 165-169] maps an analytic function $g(\xi)$ of a single complex variable ξ onto a given GASP G, i.e.,

(1.12.3) $$G(x,y) = A_u(g(\xi)) = K_{(u)} \int_C g(\xi)(\zeta-\zeta^{-1})^{2u}\zeta^{-1}\, d\zeta$$

where

$$\xi = x+iy\ (\zeta+\zeta^{-1})/2\ ,\ C = \{e^{i\phi}: 0 \le \phi \le \pi\}$$

and

$$K_{(u)} = \frac{4\,\Gamma(2u)}{(4i)^{2u}\,(\Gamma(u))^2}\ .$$

The function $g(\xi)$ is called A_u-associate of the GBSP G. The A_u-associate of the GASP G, defined by (1.12.2), is given by ([34 , p. 173])

$$g(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(2u)\ \Gamma(n+1)}\, b_n\, \xi^n.$$

Further, the inverse integral operator A_u^{-1} maps a GASP G(x,y) onto its A_u-associate $g(\xi)$ and is given by ([34, p. 173]),

(1.12.4) $$g(\xi)=A_u^{-1}(G) = \int_0^{\pi} G(r,\theta)\, K(\xi/r,\theta)\,d\theta,\quad |\xi| < r,$$

where

$$K(\xi/r,\theta) = \frac{u\ \Gamma(2u)}{2^{2u-1}(\Gamma(u+1/2))^2} \frac{(\sin\theta)^{2u}(1-\xi^2/r^2)}{(1-2(\xi/r)\cos\theta+\xi^2/r^2)^{u+1}} .$$

A GASP G, given by (1.12.2), is said to be entire if the series on the right hand side of (1.12.2) converges uniformly on the compact subsets of the whole plane. For an entire GASP G, set

$$M(r,G) = \max_{0 \le \theta \le 2\pi} |G(r,\theta)| .$$

The order $\rho_\infty(G)$ of an entire GASP G is defined as ([25], [33])

(1.12.5) $$\rho_\infty(G) = \limsup_{r\to\infty} \frac{\log\log M(r,G)}{\log r} ,$$

and, further, if $0 < \rho_\infty(G) < \infty$, then the type $T_\infty(G)$ of G is defined as

(1.12.6) $$T_\infty(G) = \limsup_{r\to\infty} \frac{\log M(r,G)}{r^{\rho_\infty(G)}} .$$

Analogous to the lower type of an entire function we define the lower type $t_\infty(G)$ of an entire GASP G, having order $\rho_\infty(G)$, $0 < \rho_\infty(G) < \infty$, as

(1.12.7) $$t_\infty(G) = \liminf_{r\to\infty} \frac{\log M(r,G)}{r^{\rho_\infty(G)}} .$$

The characterizations of order $\rho_\infty(G)$ and type $T_\infty(G)$ of an entire GASP G, in terms of the coefficients b_n in the untra-spherical harmonic expansion (1.12.2) of G, were obtained by Fryant [25] and Gilbert [34], respectively, with the help of

Gilbert's A_u-operator and inverse operator A_u^{-1}, given by (1.12.3 and (1.12.4). Thus, Gilbert [34, p. 188] showed that if the entire GASP G, given by (1.12.2), is of order $\rho_\infty(G)$, $0 < \rho_\infty(G) < \infty$ and type $T_\infty(G)$, then

$$(1.12.8) \qquad e\,\rho_\infty(G)\; T_\infty(G) = \limsup_{n \to \infty} n|b_n|^{\rho_\infty(G)/n}.$$

Further, Fryant [25] showed that the order $\rho_\infty(G)$ of an entire GASP G, given by (1.12.2), satisfies

$$(1.12.9) \qquad \rho_\infty(G) = \limsup_{n \to \infty} \frac{n \log n}{-\log |b_n|}.$$

We now give some results of McCoy [67], which depict the influence of the growth of an entire GASP G on its degree of approximation.

A GASP G, given by (1.12.2), is said to be regular in the unit disc $D_1 = \{(x,y) : x^2+y^2 < 1\}$ if the series on the right hand side of (1.12.2) converges uniformly on the compact subsets of D_1. Let $\tilde{G}$ be the class of all GASP's G regular in D_1 and continuous on $\bar{D}_1$, the closure of D_1. For $G \in \tilde{G}$, the uniform norm is defined as

$$(1.12.10) \qquad ||G|| = \max_{x^2+y^2 \le 1} |G(x,y)|$$

and the error $\Delta_n(G)$ in approximating the GASP $G \in \tilde{G}$ by GASP polynomials of degree atmost n in uniform norm is defined as

$$(1.12.11) \qquad \Delta_n(G) = \inf_{g \in \pi_n^*} ||G-g||.$$

To study the influence of the growth of an entire GASP on the approximation error $\Delta_n(G)$, given by (1.12.11), McCoy [67] has developed an invertible integral operator that is an alternate to Gilbert's A_u-operator. Thus, for a GASP G, regular about the origin, he considers the following expansion

$$(1.12.12) \quad G(r,\theta) = \sum_{n=0}^{\infty} b_n^* r^n P_n^{(\tilde{u},\tilde{u})}(\cos\theta)/P_n^{(\tilde{u},\tilde{u})}(1)$$

where $\tilde{u} = u-1/2$ and $P_n^{(\tilde{u},\tilde{u})}(t)$ are symmetric Jacobi polynomials [122, Chapter IV]. Since, it is known [122,p. 81] that

$$C_n^u(x) = \frac{\Gamma(n+2u)}{\Gamma(n+1)\ \Gamma(2u)} P_n^{(\tilde{u},\tilde{u})}(x)/P_n^{(\tilde{u},\tilde{u})}(1)$$

we get

$$b_n = \frac{\Gamma(n+1)\ \Gamma(2u)}{\Gamma(n+2u)} b_n^*, \quad n = 0,1,2,\ldots$$

where b_n's are the coefficients in the expansion (1.12.2) of G. Then the integral operator $W_{\tilde{u}}$ maps an analytic function

$$\tilde{g}(\zeta) = \sum_{n=0}^{\infty} b_n^* \zeta^n,$$

called the $W_{\tilde{u}}$-associate of the GASP G, given by (1.12.12), onto G and is defined as [67]

$$(1.12.13) \quad G(x,y) = W_{\tilde{u}}(\tilde{g}(\zeta)) = \int_0^{\pi} \tilde{g}(\zeta) \frac{\sqrt{\pi}\,\Gamma(u)}{\Gamma(u+1/2)} (\sin\phi)^{2u-1}\, d\phi$$

where $\zeta = x + iy\cos\phi$. The inverse integral operator $W_{\tilde{u}}^{-1}$ mapping the GASP G onto its $W_{\tilde{u}}$-associate $\tilde{g}(\zeta)$ is defined as

$$(1.12.14)\ \tilde{g}(\zeta) = W_{\tilde{u}}^{-1}(G) = \int_{t=-1}^{t=1} \tilde{K}_{\tilde{u}}(\zeta r^{-1},t)\, G(rt, r\sqrt{1-t^2})(1-t^2)^u\, dt$$

where

$$\tilde{K}_{\tilde{u}}(\zeta r^{-1},t) = \sum_{n=0}^{\infty} \zeta^n P_n^{(\tilde{u},\tilde{u})}(t)\, P_n^{(\tilde{u},\tilde{u})}(1)/r^n\, h_n^{(\tilde{u},\tilde{u})} ,$$

$$h_n^{(\tilde{u},\tilde{u})} = \frac{2^{2\tilde{u}+1}\,(\Gamma(n+\tilde{u}+1))^2}{(2n+2\tilde{u}+1)\;\Gamma(n+1)\;\Gamma(n+2\tilde{u}+1)} , \; n = 0,1,2,\ldots .$$

Using the integral operator $W_{\tilde{u}}$ and the inverse integral operator $W_{\tilde{u}}^{-1}$, given by (1.12.13) and (1.12.14), respectively, McCoy [67] has shown that $G \varepsilon \tilde{G}$ has an analytic continuation as an entire GASP, if and only if,

$$(1.12.15) \qquad \lim_{n \to \infty} (\Delta_n(G))^{1/n} = 0.$$

Further, $G \varepsilon \tilde{G}$ has an analytic continuation as an entire GASP of finite order $\rho_\infty(G)$, if and only if,

$$(1.12.16) \qquad \rho_\infty(G) = \limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_n(G)} < \infty$$

McCoy has also shown that $G \varepsilon \tilde{G}$ has an analytic continuation as an entire GASP of order $\rho_\infty(G)$, $0 < \rho_\infty(G) < \infty$, and finite type $T_\infty(G)$, if and only if

$$(1.12.17) \qquad \limsup_{n \to \infty} n(\Delta_n(G))^{\rho_\infty(G)/n}$$

is finite.

Some more results concerning growth and approximation of GASP's can be found in [26], [28], [33], [69] etc.

1.13. The generalized biaxisymmetric potentials (GBSP's) F are the solutions of the partial differential equation

$$(1.13.1) \qquad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{2u+1}{y}\frac{\partial F}{\partial y} + \frac{2v+1}{x}\frac{\partial F}{\partial x} = 0, \; u,v > -1/2,$$

that are even in x and y. When 2u+1 and 2v+1 are positive integers, the GBSP F is a solution of the 2u+2v+4 dimensional Laplace equation (1.11.1) that depends only on the variables

$$x = (x_1^2+x_2^2+\ldots+x_{2u+2}^2)^{1/2}, \; y = (x_{2u+3}^2+\ldots+x_{2u+2v+4}^2)^{1/2}.$$

A polynomial of degree n which is even in x and y and satisfies (1.13.1) is said to be a GBSP polynomial of degree n. Let π_n^0 denote the class of all GBSP polynomials of degree atmost 2n.

A GBSP F, regular about the origin, can be expanded as ([34, p. 170])

$$(1.13.2) \qquad F(x,y) \equiv F(r,\theta) = \sum_{n=0}^{\infty} c_n r^{2n} P_n^{(u,v)} (\cos 2\theta)$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $P_n^{(u,v)}$ are Jacobi polynomials [122, Chapter IV].

A GBSP F, given by (1.13.2), is said to be entire if the series on the right hand side of (1.13.2) converges uniformly on compact subsets of the whole plane. Set

$$M(r,F) = \max_{0 \le \theta \le 2\pi} |F(r,\theta)|.$$

Then, the order $\rho_\infty(F)$ of an entire GBSP F is defined as ([29], [68])

$$(1.13.3) \qquad \rho_\infty(F) = \limsup_{r\to\infty} \frac{\log\log M(r,F)}{\log r}$$

and, further if $0 < \rho_\infty(F) < \infty$, the type $T_\infty(F)$ of F is defined as

$$(1.13.4) \qquad T_\infty(F) = \limsup_{r \to \infty} \frac{\log M(r,F)}{r^{\rho_\infty(F)}} .$$

We define the lower type $t_\infty(F)$ of an entire GBSP F, having order $\rho_\infty(F)$, $0 < \rho_\infty(F) < \infty$, as

$$(1.13.5) \qquad t_\infty(F) = \liminf_{r \to \infty} \frac{\log M(r,F)}{r^{\rho_\infty(F)}} .$$

Recently, Fryant [29] has obtained characterizations of order $\rho_\infty(F)$ and type $T_\infty(F)$ of an entire GBSP F, in terms of the coefficients c_n in the expansion (1.13.2) of F. Thus, it has been shown that if the entire GBSP F, given by (1.13.2), is of order $\rho_\infty(F)$, then

$$(1.13.6) \qquad \rho_\infty(F) = 2 \limsup_{n \to \infty} \frac{n \log n}{-\log |c_n|}$$

and, further, if $0 < \rho_\infty(F) < \infty$, the type $T_\infty(F)$ of F is given by

$$(1.13.7) \qquad T_\infty(F) = \frac{2}{e\,\rho_\infty(F)} \limsup_{n \to \infty} n|c_n|^{\rho_\infty(F)/(2n)} .$$

McCoy ([68], [70]) has studied the influence of the growth of an entire GBSP on its degree of approximation. We now give the results in this direction.

A GBSP F, given by (1.13.2), is said to be regular in the unit disc $D_1 = \{(x,y) : x^2+y^2 < 1\}$ if the series on the right hand side of (1.13.2) converges uniformly on compact subsets of D_1. Let $\tilde{F}_\delta$, $1 \leq \delta < \infty$, be the class of all GBSP's F regular in D_1 and satisfying

$$||F||_\delta = \left(\iint_{D_1} |F(x,y)|^\delta \, dx\, dy\right)^{1/\delta} < \infty, \; 1 \le \delta < \infty$$

and let $\tilde{F}_\infty$ be the class of all GBSP's F regular in D_1 and continuous on $\bar{D}_1$, the closure of D_1, and for $F \in \tilde{F}_\infty$, let

$$||F||_\infty = \sup_{0 \le \theta \le 2\pi} |F(1,\theta)|.$$

Then $||\cdot||_\delta$, $1 \le \delta \le \infty$, is a norm on $\tilde{F}_\delta$ and is called the L^δ-norm. For $F \in \tilde{F}_\delta$, $1 \le \delta \le \infty$, the error $\Delta_{n,\delta}(F)$ in approximating the GBSP F by GBSP polynomials of degree atmost 2n, is defined as

$$(1.13.8) \qquad \Delta_{n,\delta}(F) = \inf_{g \in \pi_n^o} ||F-g||.$$

McCoy ([68,70]) has developed an invertible integral operato to measure the rate of decay of the approximation error $\Delta_{n,\delta}(F)$, $1 \le \delta \le \infty$, given by (1.13.8), of an entire GBSP F. Thus, he considered the following expansion for a GBSP F, regular about origin

$$(1.13.9) \qquad F(r,\theta) = \sum_{n=0}^{\infty} c_n^* \, r^{2n} \, P_n^{(v,u)}(\cos 2\theta)/P_n^{(v,u)}(1), \; v > u.$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $P_n^{(v,u)}$ are Jacobi polynomials. Then the integral operator $K_{v,u}$ maps an even analytic function

$$f(\zeta) = \sum_{n=0}^{\infty} c_n^* \, \zeta^{2n}$$

called the associated function of the GBSP F, given by (1.13.9), onto the GBSP F and is defined as

$$(1.13.10) \quad F(x,y) = K_{v,u}(f) = \int_0^1 \int_0^\pi f(\zeta)\, \gamma_{v,u}(1-t^2)^{v-u-1} t^{2u+1} \times$$
$$\times (\sin s)^{2v}\, ds\, dt$$

where

$$\zeta^2 = x^2 - y^2 t^2 + i\, 2xyt \cos s, \quad \gamma_{v,u} = \frac{2\Gamma(v+1)}{\Gamma(1/2)\,\Gamma(v-u)\,\Gamma(u+1/2)}.$$

The inverse operator $K^{-1}_{v,u}$ mapping the GBSP F onto its associated function $f(\zeta)$ is defined as

$$(1.13.11) \quad f(\zeta) = K^{-1}_{v,u}(F) =$$
$$\int_{-1}^{+1} F(r\xi, r\sqrt{1-\xi^2})\; S_{v,u}(\zeta^2/r^2, \xi)(1-\xi)^v (1+\xi)^u\, d\xi ,$$

where

$$S_{v,u}(\tau,\xi) = \eta_{v,u} \frac{1-\gamma}{(1+\tau)^{u+v+2}} \times {}_2F_1\left(\frac{u+v+2}{2};\ \frac{u+v+3}{2}; u+1;\ \frac{2\tau(1+\xi)}{(1+\tau)^2}\right)$$

and

$$\eta_{v,u} = \Gamma(u+v+2)/2^{u+v+1}\,\Gamma(u+1)\,\Gamma(v+1).$$

Using the integral operator $K_{v,u}$ and the inverse integral operator $K^{-1}_{v,u}$, McCoy [68, 70] has studied the rate of decay of the approximation error $\Delta_{n,\delta}(F)$, $1 \le \delta \le \infty$, given by (1.13.8), of an entire GBSP F. However, his results contain some minor errors. We give here the correct version of his results.

A GBSP $F \in \tilde{F}_\delta$, $1 \le \delta \le \infty$, has an analytic continuation as an entire GBSP, if and only if,

$$(1.13.12) \qquad \lim_{n\to\infty} (\Delta_{n,\delta}(F))^{1/n} = 0,$$

and $F \in \tilde{F}_\delta$, $1 \leq \delta \leq \infty$, has an analytic continuation as an entire GBSP of finite order $\rho_\infty(F)$, if and only if,

$$\rho_\infty(F) = 2 \limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_{n,\delta}(F)} < \infty.$$

Further, $F \in \tilde{F}_\delta$, $1 \leq \delta \leq \infty$, has an analytic continuation as an entire GBSP of order $\rho_\infty(F)$, $0 < \rho_\infty(F) < \infty$, and nonzero finite type $T_\infty(F)$, if and only if,

$$W_\delta = \limsup_{n \to \infty} n(\Delta_{n,\delta}(F))^{\rho_\infty(F)/(2n)}$$

satisfies $0 < W_\delta < \infty$, and $W_\delta = e\,\rho_\infty(F)\,T_\infty(F)/2$ also holds.

Some more results concerning the growth and approximation of GBSP's can be found in [29], [68].

1.14. In the previous sections we have discussed only those results which are relevant to our present study.

The results of Sections 1.12 and 1.13 demonstrate recent interest of workers in investigating the interrelations between the growth of entire GASP's and entire GBSP's and their degree of approximation. However, no attempt seems to have been made to study the influence of the growth of a GASP or GBSP, which is regular in a finite region, on its degree of approximation. Moreover, in general, the influence of the growth of harmonic functions in R^n, $n \geq 3$, whether entire or regular in a finite region, on their degree of approximation has not been studied at all. In the present study we have investigated these problems. In the process of our investigations we have also obtained some

new and fairly general results concerning the growth of analytic functions and the growth of entire functions.

In the present thesis, the rate of decay of the degree of approximation of analytic functions, harmonic functions in R^n, $n \geq 3$, GASP's and GBSP's has been studied in terms of the growth of their maximum moduli. The thesis consists of seven chapters, Chapter 1 being the introduction. In Chapters 2,3 and 6, result concerning the growth of functions analytic in a finite disc and that of entire functions are developed that are needed in Chapters 4,5 and 7 for studying the approximation of analytic functions, harmonic functions in R^n, $n \geq 3$, GASP's and GBSP's.

In Chapter 2 we have introduced the concepts of (α,β)-order and lower (α,β)-order for a function analytic in a finite disc D_R, $0 < R < \infty$. The concept of (α,β)-order introduced here improves an analogous concept due to Šeremeta [92]. Characterizations of (α,β)-order and lower (α,β)-order of analytic functions have been obtained in terms of their Taylor coefficients in this chapter. The results found here generalize and improve several known results. The results of this chapter find applications in the sequel in the study of growth and approximation of harmonic functions in R^n, $n \geq 3$, GASP's and GBSP's, regular in a finite region.

In Chapter 3, we consider analytic functions in a finite disc D_R, $0 < R < \infty$, that are of slow growth. For a precise measure of growth of such functions the concepts of α-logarithm order, lower α-logarithmic order, (α,α_*)-logarithmic type and lower (α,α_*)-logarithmic type are introduced. The coefficient equivalents of these growth parameters have been found in this chapter. Some of our results in this chapter generalize the results in [54]. The results obtained here also find useful applications in subsequent chapters for a precise study of growt and approximation of harmonic functions in R^n, $n \geq 3$, GASP's and GBSP's regular in a finite region.

Chapter 4 deals with the growth and approximation of harmonic functions regular in a finite hyperball in R^n, $n \geq 3$. We first introduce the concepts of generalized growth parameters for such functions. The rates of decay of the degree of approximation in L^δ-norm, $1 \leq \delta \leq \infty$, of the harmonic functions, regular in a finite hyperball in R^n, $n \geq 3$, have been studied in terms of the generalized growth parameters. Further, the interrelation between the coefficients in the hyperspherical harmonic expansion of harmonic functions, regular in a finite hyperball in R^n,$n \geq 3$, and their generalized growth parameters have been obtained.

In Chapter 5, we have considered the growth and approximatior of GASP's and GBSP's, regular in a finite disc. For this purpose some new concepts of generalized growth parameters have been introduced separately for GASP's and GBSP's, regular in a finite

disc. The interrelations between the degree of approximation in L^{δ}-norm, $1 \leq \delta \leq \infty$, of a GASP or a GBSP, regular in a finite disc, and their generalized growth parameters have been investigated in this chapter. Further, the coefficients in the ultraspherical harmonic expansion of a GASP, regular in a finite disc, have been studied in terms of its generalized growth parameters.

The study in Chapters 4 and 5 seems to be new in the context of harmonic functions in R^n, regular in a finite hyperball, and GASP's and GBSP's regular in a finite disc.

Chapters 6 and 7 are devoted to the study of growth and approximation of entire functions, entire harmonic functions in R^n, $n \geq 3$, entire GASP's and entire GBSP's.

In Chapter 6, we introduce the concepts of generalized (α,α)-order and generalized lower (α,α)-order for an entire function. These growth parameters measure satisfactorily the growth of a class of entire functions of slow growth for which the corresponding concepts of generalized (α,β)-order and generalized lower (α,β)-order due to Šeremeta [93] and Shah [102] fail to give any specific information. Coefficient characterizations of generalized (α,α)-order and generalized lower (α,α)-order have also been obtained in this chapter. The results of this chapter include several known results and find useful applications in Chapter 7.

In Chapter 7, we first consider the approximation of entir functions over a Caratheodory domain in L^{δ}-norm, $1 \leq \delta < \infty$. Approximation of entire functions has also been considered in uniform norm over a compact set having nonzero transfinite diameter. Further, we study the approximation of entire harmon: functions in R^n, $n \geq 3$, over a finite hyperball, in L^{δ}-norm, $1 \leq \delta \leq \infty$. Finally, in this chapter, the approximation of an entire GASP or an entire GBSP on a finite disc in L^{δ}-norm, $1 \leq \delta \leq \infty$, has been studied. The results of this chapter extend improve, and generalize some results of Giroux [35] and McCoy [67,68,70].

In Chapter 7, we first consider the approximation of enti functions over a Caratheodory domain in L^{δ}-norm, $1 \leq \delta < \infty$. Approximation of entire functions has also been considered in uniform norm over a compact set having nonzero transfinite diameter. Further, we study the approximation of entire harmon functions in R^{n}, $n \geq 3$, over a finite hyperball, in L^{δ}-norm, $1 \leq \delta \leq \infty$. Finally, in this chapter, the approximation of an entire GASP or an entire GBSP on a finite disc in L^{δ}-norm, $1 \leq \delta \leq \infty$, has been studied. The results of this chapter exten improve, and generalize some results of Giroux [35] and McCoy [67,68,70].

CHAPTER 2

GENERALIZED ORDERS OF FUNCTIONS ANALYTIC IN A FINITE DISC I

2.1. Let L^{O} be the class of functions $h(x)$ satisfying the following conditions (H,i) and (H,ii) :

(H,i) $h(x)$ is defined on $[a,\infty)$ and is positive, continuous, strictly increasing and tends to ∞ as $x \to \infty$.

(H,ii) $\lim_{x \to \infty} h(x(1+\delta(x)))/h(x) = 1$ for every function $\delta(x)$ such that $\delta(x) \to 0$ as $x \to \infty$.

A function $h(x) \varepsilon L^{O}$ is said to belong to the class Λ if it is slowly increasing [91] , i.e., if the following stronger condition (H,iii) holds in place of (H,ii)

(H,iii) $\lim_{x \to \infty} h(cx)/h(x) = 1$ for all c, $0 < c < \infty$.

We observe that $L_{*}^{O} \subset L^{O}$ and $\Lambda_{*} \subset \Lambda$, since the functions of the classes L_{*}^{O} and Λ_{*} defined in Section 1.3, are taken to be differentiable on $[a,\infty)$, while the functions in the classes L^{O} and Λ need only to be continuous on $[a,\infty)$.

In the rest of the work, wherever necessary, we shall assume that $h(x) \varepsilon L^{O}$ has been extended over $(-\infty,a)$ by the definition $h(x) = h(a)$ for $x \varepsilon (-\infty,a)$.

We have the following definition :

DEFINITION 2.1.1. <u>A function</u> $f(z)$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>said to be of</u> (α,β)-<u>order</u> $\rho(\alpha,\beta,f)$ <u>and lower</u> (α,β)-<u>order</u> $\lambda(\alpha,\beta,f)$, $0 \leq \lambda(\alpha,\beta,f) \leq \rho(\alpha,\beta,f) \leq \infty$, <u>if</u>

$$(2.1.1) \qquad \begin{matrix}\rho(\alpha,\beta,f)\\ \lambda(\alpha,\beta,f)\end{matrix} = \lim_{r \to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\beta(R/(R-r))},$$

<u>where</u>, $M(r,f)$ <u>is the maximum modulus of</u> $f(z)$ <u>for</u> $|z| = r < R$; $\alpha(x) \in \Lambda$ <u>and</u> $\beta(x) \in L^O$ <u>satisfy either of the conditions</u> (2.1.2) <u>and</u> (2.1.3) :

$$(2.1.2) \qquad \alpha(x) = \beta(x) = \log x$$

$$(2.1.3) \qquad \lim_{x \to \infty} \frac{\alpha(x/F(x,c))}{\alpha(x)} = 1 \quad \text{for every } c,\ 0 < c < \sigma,$$

<u>where</u> $F(x,c) = \beta^{-1}(c\alpha(x))$, $\sigma = \infty$ <u>if</u> $\alpha \not\equiv \beta$ <u>and</u> $\sigma = 1$ <u>if</u> $\alpha \equiv \beta$.

Our notion of (α,β)-order $\rho(\alpha,\beta,f)$ is a refinement of the corresponding notion of Šeremeta [92] , since in Definition 2.1.1. $\beta(x) \in L^O$, while in (1.9.19) Šeremeta has assumed that $\beta(x)$ is restricted to the class Λ_*.

It is seen that $\alpha(x) = \log_{p+q} x$ and $\beta(x) = (\log_q x)^d$, $p \geq 1$, $q \geq 0$ and $0 < d < \infty$ satisfy (2.1.3). Thus, with the choices $\alpha(x) = \log_p x$, $p \geq 1$, and $\beta(x) = \log x$ or $\alpha(x) = \log_p x$, $p \geq 1$, and $\beta(x) = x^d$, $0 < d < \infty$, growth parameters of Kapoor and Gopal ([53, 55]) for analytic functions of fast growth follow as particular cases of our growth parameters. We note that the choice $\alpha(x) = \log_p x$, $p \geq 1$, and $\beta(x) = x^d$, $0 < d < \infty$, is not permissible in Šeremeta's definition (1.9.19), corresponding to (α,β)-order.

We now give some other choices of $\alpha(x)$ and $\beta(x)$ which satisfy (2.1.3).

(i) $\alpha(x) = \log x$ and $\beta(x) = (\log x)^d$, $1 < d < \infty$.

(ii) $\alpha(x) = \log_p x$ and $\beta(x) = (\log_p x)^d$, $p \geq 2$, $1 \leq d < \infty$.

(iii) $\alpha(x) = (\log x)^d$ or $\alpha(x) = (\log_p x)^{d'}$ and $\beta(x)=\exp(\log$
$0 < d < 1$, $0 < d' < \infty$, $p \geq 2$.

(iv) $\alpha(x) = \log x$ and $\beta(x) = \exp (\log_2 x)^d$, $1 < d < \infty$.

We note that (2.1.3) is not satisfied if $\alpha(x) = \beta(x) = \log x$ or $\alpha(x) = \exp (\log x)^d$ and $\beta(x) = \exp (K(\log x)^d)$, $0 < d < 1$, $1 < K < \infty$.

DEFINITION 2.1.2. *A function* $f(z)$, *analytic in* D_R, $0 < R < \infty$, *and having* (α,β)-*order* $\rho(\alpha,\beta,f)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$ *is said to be of regular* (α,β)-*growth if* $\lambda(\alpha,\beta,f) = \rho(\alpha,\beta,f)$ *an* $f(z)$ *is said to be of irregular* (α,β)-*growth if it is not of regular* (α,β)-*growth.*

Since, for the case $\alpha(x) = \beta(x) = \log x$, $\rho(\alpha,\beta,f)$ and $\lambda(\alpha,\beta,f)$ have been extensively studied by various workers ([8], [13],[50, 51, 52], [56], [65], [110] etc.) *we shall confine ourselves in Sections* 2.2 *to* 2.5 *to the case that* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (2.1.3).

In Section 2.2, we first obtain a characterization of the class L^o. In this section, characterizations of (α,β)-order and lower (α,β)-order, in terms of the maximum term and the central index of an analytic function, have also been obtained.

Characterizations of (α,β)-order of an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, $a_n \neq 0$ for all n, have been found in Section 2.3 in terms of the coefficients a_n. Section 2.4 deals with the coefficient characterizations of lower (α,β)-order. In Section 2.5, we first obtain a necessary condition for an analytic function to be of regular (α,β)-growth. A decomposition theorem for an analytic function of irregular (α,β)-growth has also been proved in this section.

The following notations and conventions are used throughout the rest of the work.

NOTATIONS AND CONVENTIONS.

(I) $$P(\theta) = \begin{cases} \theta & \text{if } \alpha(x) = \beta(x) = \log x \text{ or } \alpha \not\equiv \beta \\ \max(1,\theta) & \text{if } \alpha \equiv \beta (\alpha(x) \neq \log x) \end{cases}$$

(II) $$\tilde{P}(\theta) = \begin{cases} P(\theta) & \text{if } \alpha \not\equiv \beta \text{ or } \alpha \equiv \beta\ (\alpha(x) \neq \log x) \\ \max(1,\theta) & \text{if } \alpha(x) = \beta(x) = \log x. \end{cases}$$

In (I) and (II), $\theta \equiv \theta(\alpha,\beta)$ is a function of $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$.

(III) If $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$ and A,B are finite positive numbers, then for $x \leq 1$, the quotient

$$\frac{\alpha(A)}{\beta(B/\log^+ x)}$$

will be interpreted to be equal to zero.

To avoid some trivial cases we shall assume throughout the rest of the chapter that $M(r,f) \to \infty$ as $r \to R$.

2.2. In this section we obtain two lemmas that are needed in the sequel. Lemma 2.2.1 gives a characterization of the class of functions L°. In Lemma 2.2.2, characterizations of (α,β)-order $\rho(\alpha,\beta,f)$ and lower (α,β)-order $\lambda(\alpha,\beta,f)$ have been obtained in terms of the maximum term and the central index of the analytic function $f(z)$.

LEMMA 2.2.1. *Let $h(x)$, defined on $[a,\infty)$, be positive, continuous, strictly increasing and tending to ∞ as $x \to \infty$. Then, the following conditions (i) and (ii) are equivalent.*

$$\text{(i)} \quad \lim_{c \to 1^-} \{\liminf_{x \to \infty} \frac{h(cx)}{h(x)}\} = 1$$

$$\text{(ii)} \quad \lim_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} = 1,$$

for every function $\delta(x)$ such that $\delta(x) \to 0$ as $x \to \infty$.

PROOF. (a) Assume that $h(x)$ satisfies (i). Then, given $\varepsilon > 0$, there exists $c_o(\varepsilon)$ such that

$$\liminf_{x \to \infty} \frac{h(cx)}{h(x)} \geq 1-\varepsilon \tag{2.2.1}$$

for all c, $c_o(\varepsilon) \leq c < 1$.

Now, first let $\delta(x) \geq 0$. Then

$$\liminf_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \geq 1, \tag{2.2.2}$$

since $h(x)$ is a strictly increasing function of x. Set $C_o(\varepsilon) = 1/c_o(\varepsilon)$. Then, since $\delta(x) \to 0$ as $x \to \infty$, we have

$$1 + \delta(x) \leq C_o(\varepsilon)$$

for all $x > x_o(C_o(\varepsilon))$. Thus

$$\limsup_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \leq \limsup_{x \to \infty} \frac{h(C_o(\varepsilon)x)}{h(x)}$$

$$= 1/(\liminf_{y \to \infty} \frac{h(c_o(\varepsilon)y)}{h(y)}),$$

where $y = C_o(\varepsilon)\ x$. Using (2.2.1), the above inequality gives that

$$\limsup_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \leq \frac{1}{1-\varepsilon}$$

and since $\varepsilon > 0$ is arbitrary we get

$$\limsup_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \leq 1. \tag{2.2.3}$$

Combining (2.2.2) and (2.2.3) we see that (ii) holds for the case $\delta(x) \geq 0$.

Next, let $\delta(x) \leq 0$. Then

$$\limsup_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \leq 1. \tag{2.2.4}$$

Since $\delta(x) \to 0$ as $x \to \infty$, we have $1+\delta(x) \geq c_o(\varepsilon)$ for $x > x_o(c_o(\varepsilon))$. Thus, using (2.2.1) we get

$$\liminf_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \geq \liminf_{x \to \infty} \frac{h(c_o(\varepsilon)x)}{h(x)} \geq 1-\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we get

$$\liminf_{x \to \infty} \frac{h(x(1+\delta(x)))}{h(x)} \geq 1. \tag{2.2.5}$$

In view of (2.2.4) and (2.2.5) we see that (ii) holds for the case $\delta(x) \leq 0$ also.

We have proved above that if $h(x)$ satisfies (i) then it also satisfies (ii) for the cases when $\delta(x) \geq 0$ or $\delta(x) \leq 0$. It now follows from these two particular cases that if $h(x)$ satisfies (i) then it also satisfies (ii) for a general $\delta(x)$, $\delta(x) \to 0$ as $x \to \infty$; since $\delta(x) = \delta^+(x) + \delta^-(x)$, where $\delta^+(x) \geq 0$ is given by $\delta^+(x) = \delta(x)$ if $\delta(x) \geq 0$, $\delta^+(x) = 0$ otherwise; and $\delta^-(x) \leq 0$ is given by $\delta^-(x) = \delta(x)$ if $\delta(x) \leq 0$, $\delta^-(x) = 0$ otherwise.

(b) Assume that $h(x)$ does not satisfy (i). Then

$$\lim_{c \to 1^-} \{\liminf_{x \to \infty} \frac{h(cx)}{h(x)}\} = K < 1.$$

Hence

$$\liminf_{x \to \infty} \frac{h(cx)}{h(x)} \leq K \tag{2.2.6}$$

for all c, $0 < c < 1$. Choose a strictly increasing sequence $\{s_n\}_{n=1}^{\infty}$ of positive numbers such that

$$\lim_{n \to \infty} s_n = 1.$$

Let $K < K' < 1$. Then, from (2.2.6), there exists $x_1 > 0$ such that

$$\frac{h(s_1\ x_1)}{h(x_1)} < K',$$

and, in general, for a given n, $n \geq 1$, there exists $x_{n+1} > 2x_n$ such that

$$\frac{h(s_{n+1}\ x_{n+1})}{h(x_{n+1})} < K'.$$

Clearly $x_n \to \infty$ as $n \to \infty$. We now consider a step function $\delta_o(x)$ defined as

$$\delta_o(x) = s_n - 1, \quad x_n \le x < x_{n+1}.$$

Then, $\delta_o(x) \to 0$ as $x \to \infty$. But

$$\liminf_{x \to \infty} \frac{h(x(1+\delta_o(x)))}{h(x)} \le \liminf_{n \to \infty} \frac{h(s_n\, x_n)}{h(x_n)} \le K' < 1.$$

Hence, for $\delta_o(x)$, the relation

$$\lim_{x \to \infty} \frac{h(x(1+\delta_o(x)))}{h(x)} = 1$$

does not hold and so $h(x)$ does not satisfy (ii) also.

In view of parts (a) and (b) above the proof of the lemma is complete.

COROLLARY. *Let $h(x)$, defined on $[a,\infty)$, be positive, continuous, strictly increasing and tending to ∞, as $x \to \infty$. Then, $h(x) \in L^o$, if and only if,*

$$\lim_{c \to 1^-} \{\liminf_{x \to \infty} \frac{h(cx)}{h(x)}\} = 1.$$

LEMMA 2.2.2. *Let $f(z)$ be analytic in D_R, $0 < R < \infty$, with (α,β)-order $\rho(\alpha,\beta,f)$ and lower (α,β)-order $\lambda(\alpha,\beta,f)$. Assume that the maximum term $\mu(r)$ and the central index $\nu(r)$ of $f(z)$ are unbounded functions of r. Then, for $\alpha \not\approx \beta$,*

$$(2.2.7)_a \qquad \rho(\alpha,\beta,f) = \Phi_1 = \theta_1$$

and

$$(2.2.8)_a \qquad \lambda(\alpha,\beta,f) = \Phi_2 = \theta_2$$

where

$$(2.2.9)\qquad \begin{matrix}\Phi_1\\ \Phi_2\end{matrix} = \lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \mu(r))}{\beta(R/(R-r))}$$

and

$$(2.2.10)\qquad \begin{matrix}\theta_1\\ \theta_2\end{matrix} = \lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\nu(r))}{\beta(R/(R-r))} .$$

Further, for the case $\alpha \equiv \beta$, if $\rho(\alpha,\alpha,f) \geq 1$, then

$$(2.2.7)_b \qquad \rho(\alpha,\alpha,f) = \max(1,\Phi_1) = \max(1,\theta_1)$$

and if $\lambda(\alpha,\alpha,f) \geq 1$, then

$$(2.2.8)_b \qquad \lambda(\alpha,\alpha,f) = \max(1,\Phi_2) = \max(1,\theta_2).$$

PROOF. (i) $\rho(\alpha,\beta,f) = P(\Phi_1)$; $\lambda(\alpha,\beta,f) = P(\Phi_2)$:

Using (1.8.4), for $b > 2$, we have

$$(2.2.11)\qquad \frac{R-r}{bR}\,\nu(r) \leq \nu(r)\log\frac{r+(R-r)/b}{r}$$
$$\leq \int_r^{r+(R-r)/b} \frac{\nu(t)}{t}\,dt$$
$$\leq \log \mu(r+(R-r)/b)$$

for all r sufficiently near to R. Now, (2.2.11) gives that

$$(2.2.12)\qquad \nu(r+(R-r)/b) \leq \frac{\log \mu(r + \frac{R-r}{b} + \frac{R-r}{b}(1-\frac{1}{b}))}{((R-r) - (R-r)/b)/(bR)}$$
$$\leq \frac{b^2 R}{(b-1)(R-r)} \log \mu(r + 2\frac{R-r}{b}).$$

From (1.8.5) and (2.2.12), for all r sufficiently near to R, we obtain

$$(2.2.13)\quad \log M(r,f) \le \log \mu(r) + \log \frac{R}{R-r} + \log 3 + \log \nu\left(r + \frac{R-r}{b}\right)$$

$$\le \log \mu(r) + 2 \log \frac{R}{R-r} + \log 3 + \log \frac{b^2}{b-1} +$$

$$+ \log \log \mu\left(r + 2\frac{R-r}{b}\right)$$

$$\le 5\left(\log \mu\left(r + 2\frac{R-r}{b}\right)\right) \log \frac{R}{R-r}$$

$$\le \frac{R}{R-r} \log \mu\left(r + 2\frac{R-r}{b}\right).$$

To prove $\rho(\alpha,\beta,f) \le P(\Phi_1)$, we can assume without loss of generali that $\rho(\alpha,\beta,f) > P(0)$ since there is nothing to prove if $\rho(\alpha,\beta,f) \le P(0)$. Let $P(0) < \rho(\alpha,\beta,f) < \infty$. Then, given $\varepsilon > 0$, $\rho(\alpha,\beta,f) - \varepsilon > P(0)$, there exists a sequence $\{r_k\}$, $r_k \to R$, such that

$$\log M(r_k,f) > \alpha^{-1} \left(\bar{\rho}\ \beta(R/(R-r_k))\right)$$

where $\bar{\rho} = \rho(\alpha,\beta,f) - \varepsilon$. Now, from (2.2.13) and the above relation, for all sufficiently large values of k, we have

$$\log \mu\left(r_k + 2\frac{R-r_k}{b}\right) \ge \frac{R-r_k}{R}\ \alpha^{-1} \left(\bar{\rho}\ \beta(R/(R-r_k))\right).$$

The above relation, on using (2.1.3), for all sufficiently large values of k, gives that

$$\alpha\left(\log \mu\left(r_k + 2\frac{R-r_k}{b}\right)\right) \ge \alpha\left\{\frac{R-r_k}{R}\ \alpha^{-1}\left(\bar{\rho}\ \beta(R/(R-r_k))\right)\right\}$$

$$\sim \bar{\rho}\ \beta(R/(R-r_k))$$

and so

$$(2.2.14)\qquad \Phi_1 \ge \bar{\rho} \limsup_{k \to \infty} \frac{\beta\left(\frac{R}{R-r_k}\right)}{\beta\left(\frac{R}{(1-2/b)(R-r_k)}\right)}$$

$$\geq \bar{\rho} \liminf_{k \to \infty} \frac{\beta(\frac{R}{R-r_k})}{\beta(\frac{bR}{(b-2)(R-r_k)})} .$$

Since $\beta(x) \in L^O$, using Lemma 2.2.1 and (2.2.14) we get

$$\Phi_1 \geq \bar{\rho} = \rho(\alpha,\beta,f) - \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary we finally have $\Phi_1 \geq \rho(\alpha,\beta,f)$. Thu

(2.2.15) $\qquad P(\Phi_1) \geq \rho(\alpha,\beta,f)$.

For $\rho(\alpha,\beta,f) = \infty$ the above arguments with an arbitrarily large number in place of $\bar{\rho}$ give that $\Phi_1 = \infty$.

To prove $\lambda(\alpha,\beta,f) \leq P(\Phi_2)$, we assume that $\lambda(\alpha,\beta,f) > P(O)$, since there is nothing to prove if $\lambda(\alpha,\beta,f) \leq P(O)$. Let $P(O) < \lambda(\alpha,\beta,f) < \infty$. Then, given $\varepsilon > 0$, $\lambda(\alpha,\beta,f) - \varepsilon > P(O)$, there exists $r_o = r_o(\varepsilon)$ such that, for $r > r_o$, we have

$$\log M(r,f) > \alpha^{-1} (\bar{\lambda}\ \beta(R/(R-r))),$$

where $\bar{\lambda} = \lambda(\alpha,\beta,f) - \varepsilon$. Thus, using (2.2.13), for all r sufficiently near to R, we have

(2.2.16) $$\alpha(\log \mu(r+2\tfrac{R-r}{b})) \geq \alpha\{\tfrac{R-r}{R}\ \alpha^{-1} (\bar{\lambda}\ \beta(R/(R-r)))\}$$
$$\sim \bar{\lambda}\ \beta(R/(R-r)),$$

where in the last inequality we have used (2.1.3). Since $\beta(x) \in L^O$, the relation (2.2.16), in view of Lemma 2.2.1, gives that $\Phi_2 \geq \lambda(\alpha,\beta,f) - \varepsilon$ and so, letting $\varepsilon \to 0$, we get $\Phi_2 \geq \lambda(\alpha,\beta,f)$ Thus

(2.2.17) $\qquad P(\Phi_2) \geq \lambda(\alpha,\beta,f)$.

For $\lambda(\alpha,\beta,f) = \infty$, the above arguments with an arbitrarily large number in place of $\bar{\lambda}$ give that $\Phi_2 = \infty$.

From the well known inequality $\mu(r) \leq M(r,f)$, it follows that

(2.2.18) $\qquad \Phi_1 \leq \rho(\alpha,\beta,f)$

and

(2.2.19) $\qquad \Phi_2 \leq \lambda(\alpha,\beta,f)$.

Combining (2.2.15) with (2.2.18) and (2.2.17) with (2.2.19) we see that, if $\alpha \not\equiv \beta$, then $\rho(\alpha,\beta,f) = \Phi_1$ and $\lambda(\alpha,\beta,f) = \Phi_2$. In case $\alpha \equiv \beta$, it is seen from (2.2.15) and (2.2.18) that, if $\rho(\alpha,\alpha,f) \geq 1$, then $\rho(\alpha,\alpha,f) = P(\Phi_1)$. If we take $\lambda(\alpha,\alpha,f) \geq 1$ for the case $\alpha \equiv \beta$, then it follows from (2.2.17) and (2.2.19) that $\lambda(\alpha,\alpha,f) = P(\Phi_2)$.

(ii) $\underline{P(\Phi_1) = P(\theta_1)\ ;\ P(\Phi_2) = P(\theta_2)}$:

Let $r_* > R/2$. Then, by (1.8.4), we have

$$\log \mu(r) < \log \mu(r_*) + \nu(r) \log (r/r_*)$$
$$< \log \mu(r_*) + \nu(r) \log 2$$

for all $r > r_*$. Thus, we get

(2.2.20) $\qquad \Phi_1 \leq \theta_1,\ \Phi_2 \leq \theta_2$.

Before proceeding further we note that the condition (2.1.3) implies that

$$(2.2.21)\qquad \lim_{x\to\infty} \frac{\alpha(x\,F(x,c))}{\alpha(x)} = 1, \text{ for every } c,\ 0 < c < \sigma,$$

where $\sigma = \infty$ if $\alpha \not\equiv \beta$ and $\sigma = 1$ if $\alpha \equiv \beta$. For, by (2.1.3), for all sufficiently large values of x, we have

$$\begin{aligned}\alpha(x) &= \alpha\left(\frac{x\,F(x,c)}{F(x,c)}\right)\\ &\geq \alpha\left(\frac{x\,F(x,c)}{F(x\,F(x,c),c)}\right)\\ &\sim \alpha(x\,F(x,c))\end{aligned}$$

for $0 < c < \sigma$ and

$$\alpha(x) \leq \alpha(x\,F(x,c))$$

holds trivially.

From (2.2.11), for all r sufficiently near to R, we get

$$\frac{\nu(r)}{b-1} \leq \frac{R}{R-\bar{r}} \log \mu(\bar{r})$$

where $\bar{r} = r+(R-r)/b$, $b > 2$. Now, let $\Phi_1 < \infty$. Then, for all r sufficiently near to R, using (2.2.9) and the above inequality, we have,

$$\frac{\nu(r)}{b-1} \leq \frac{R}{R-\bar{r}}\ \alpha^{-1}\ (\Phi\beta(R/(R-\bar{r}))),$$

where $\Phi = P(\Phi_1) + \varepsilon$, $\varepsilon > 0$. Applying (2.2.21), the above relation gives that

$$(2.2.22)\qquad \alpha\left(\frac{\nu(r)}{b-1}\right) \leq \alpha\left\{\frac{R}{R-\bar{r}}\ \alpha^{-1}(\Phi\ \beta(R/(R-\bar{r})))\right\}$$

$$\sim \Phi\ \beta(R/(R-\bar{r})).$$

Since $\varepsilon > 0$ is arbitrary, $\alpha(x) \in \Lambda$ and $\beta(x) \in L^{O}$, using Lemma 2.2.1, (2.2.22) gives

(2.2.23) $$\theta_1 \leq P(\Phi_1).$$

Similarly, it can be proved that

(2.2.24) $$\theta_2 \leq P(\Phi_2).$$

It now follows from (2.2.20) and (2.2.23) that $P(\theta_1) = P(\Phi_1)$ while (2.2.20) and (2.2.24) give that $P(\theta_2) = P(\Phi_2)$.

In view of parts (i) and (ii) above the proof of the lemma is complete.

REMARKS. (i) For the case $\alpha(x) = \beta(x) = \log x$, results analogous to Lemma 2.2.2 have been obtained by Sons [110] and are given by (1.8.6), (1.8.7) and (1.8.8).

(ii) We observe that if $\rho(\alpha,\alpha,f) < 1$ and $\alpha(x)$ satisfies (2.1.3) then $\rho(f) \equiv \rho(\log,\log, f) \leq 1$. Similarly, if $\lambda(\alpha,\alpha,f) < 1$ and $\alpha(x)$ satisfies (2.1.3), then $\lambda(f) \equiv \lambda(\log,\log,f) \leq 1$. Thus $(2.2.7)_b$ is not applicable for a subclass of analytic functions having order atmost one, while $(2.2.8)_b$ is not applicable for a subclass of analytic functions having lower order atmost one. However, analytic functions with order $\rho(f)$, $0 < \rho(f) \leq 1$ or analytic functions with lower order $\lambda(f)$, $0 < \lambda(f) \leq 1$ are already extensively studied ([50, 51], [56], [65], [110]) and the analytic functions of zero order will be studied in Chapter 3.

2.3. The characterizations of (α,β)-order $\rho(\alpha,\beta,f)$ of a function $f(z)$, analytic in D_R, $0 < R < \infty$, in terms of the coefficients in the Taylor expansion of $f(z)$ are obtained in this section. We prove

<u>THEOREM</u> 2.3.1. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$; <u>analytic in</u> D_R, $0 < R < \infty$ <u>be of</u> (α,β)-<u>order</u> $\rho(\alpha,\beta,f)$. <u>Then, if</u> $\alpha \not\equiv \beta$, <u>we have</u>

(2.3.1) $$\rho(\alpha,\beta,f) = P(\theta)$$

<u>where</u>

$$\theta = \limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta(\lambda_n/\log^+|a_n|R^{\lambda_n})}.$$

<u>The equation</u> (2.3.1) <u>continues to hold for</u> $\alpha \equiv \beta$ <u>provided</u> $\rho(\alpha,\alpha,f) \geq 1$.

<u>PROOF</u>. First, let $P(0) < \theta < \infty$ and $P(0) < \theta - \varepsilon$, $\varepsilon > 0$. Then, there exists a sequence $\{n_k\}$ of positive integers tending to ∞ such that

(2.3.2) $$\log |a_{n_k}| R^{\lambda_{n_k}} > \lambda_{n_k}/F(\lambda_{n_k},1/\bar{\theta})$$

where $F(\lambda_{n_k},1/\bar{\theta}) = \beta^{-1}((1/\bar{\theta})\alpha(\lambda_{n_k}))$, $\bar{\theta} = \theta - \varepsilon$. We choose a sequence $\{r_k\}$ defined as

$$R/r_k = \exp \{1/(b\ F(\lambda_{n_k},1/\bar{\theta}))\},$$

where b is a constant such that $1 < b < \infty$. Using (2.3.2) and Cauchy's inequality, for the sequence $\{r_k\}$, we have

$$\log M(r_k,f) \geq \log |a_{n_k}| + \lambda_{n_k} \log r_k$$

$$\geq \lambda_{n_k}/F(\lambda_{n_k}, 1/\bar{\theta}) - \lambda_{n_k} \log (R/r_k)$$

$$= (b-1)\, \lambda_{n_k} \log (R/r_k)$$

$$= (b-1) \log (R/r_k)\, \alpha^{-1} (\bar{\theta}\ \beta(\frac{1}{b \log (R/r_k)})).$$

In view of the condition (2.1.3), the above relation, as $k \to \infty$, gives that

$$\text{(2.3.3)} \quad \alpha(\frac{b}{b-1} \log M(r_k,f)) \geq \alpha(b \log (R/r_k)\alpha^{-1} (\bar{\theta}\ \beta(\frac{1}{b \log (R/r_k)}))$$

$$\sim \bar{\theta}\ \beta(\frac{1}{b \log (R/r_k)}).$$

Since $\beta(x) \in L^0$, as $k \to \infty$, we have

$$\text{(2.3.4)} \qquad \beta(\frac{1}{\log (R/r_k)}) \sim \beta(\frac{R}{R-r_k}).$$

Now, (2.3.3) and (2.3.4) together give that

$$\limsup_{k \to \infty} \frac{\alpha(\frac{b}{b-1}\log M(r_k,f))}{\beta(\frac{R}{R-r_k})} \geq \bar{\theta} \liminf_{k \to \infty} \frac{\beta(\frac{1}{b \log (R/r_k)})}{\beta(\frac{1}{\log (R/r_k)})}.$$

Since $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, applying Lemma 2.2.1, the above relation yields that $\rho(\alpha,\beta,f) \geq \bar{\theta} = \theta - \varepsilon$. Thus, letting $\varepsilon \to 0$, we get $\rho(\alpha,\beta,f) \geq \theta$ and so

$$\text{(2.3.5)} \qquad \rho(\alpha,\beta,f) \geq P(\theta).$$

Since, by the hypothesis of the theorem $\rho(\alpha,\beta,f) \geq P(0)$, (2.3.5) obviously holds for $\theta \leq P(0)$. For $\theta = \infty$ the above analysis, with an arbitrarily large number in place of $\bar{\theta}$, gives that $\rho(\alpha,\beta,f) = \infty$.

To prove the reverse inequality in (2.3.5), first let $\theta <$ ·
Then, for $n > n_o = n_o(\varepsilon)$, $\varepsilon > 0$, we have

$$(2.3.6) \qquad |a_n|\, R^{\lambda_n} \leq \exp\,(\lambda_n/F(\lambda_n, 1/\theta'))$$

where $F(\lambda_n, 1/\theta') = \beta^{-1}(1/\theta')\ \alpha(\lambda_n))$ and $\theta' = P(\theta) + \varepsilon$. We now find a natural number $n(r)$, for every $r < R$, such that

$$\lambda_{n(r)} \leq \alpha^{-1}\ (\theta'\ \beta(\frac{1}{\log\,(R/r)})) < \lambda_{n(r)+1}.$$

For $n > n(r)$, we have

$$(2.3.7) \qquad |a_n|\, r^{\lambda_n} \leq \exp\,\{\frac{\lambda_n}{F(\lambda_n, 1/\theta')} + \lambda_n \log\,(r/R)\}$$

$$\leq \exp\,\{\lambda_n \log\,(R/r) + \lambda_n \log\,(r/R)\} = 1.$$

For $n_o \leq n \leq n(r)$, by (2.3.6), we obtain

$$(2.3.8) \qquad |a_n|\, r^{\lambda_n} \leq |a_n|\, R^{\lambda_n}$$

$$\leq \exp\,(\lambda_n/F(\lambda_n, 1/\theta'))$$

$$\leq \exp\,(\lambda_{n(r)})$$

$$\leq \exp\,(\alpha^{-1}\ (\theta'\ \beta(\frac{1}{\log\,(R/r)}))).$$

From (2.3.7) and (2.3.8), we get

$$(2.3.9) \qquad \mu(r) \leq \exp\,(\alpha^{-1}\ (\theta'\ \beta\,(\frac{1}{\log\,(R/r)})))$$

for all r sufficiently near to R. We can assume without loss of generality that $\mu(r) \to \infty$ as $r \to R$ since otherwise $\{|a_n|\, R^{\lambda_n}\}$ is bounded and so $\theta \leq P(0)$ and $\rho(\alpha,\beta,f) \leq P(0)$. Now, (2.3.9) gives that

$$\frac{\alpha(\log \mu(r))}{\beta(R/(R-r))} \le \frac{\alpha(\log \mu(r))}{\beta(\frac{1}{\log (R/r)})} \le \theta'.$$

The above relation, in view of Lemma 2.2.2, yields that $\rho(\alpha,\beta,f) \le P(\theta) + \varepsilon$ and so, letting $\varepsilon \to 0$, we have

(2.3.10) $$\rho(\alpha,\beta,f) \le P(\theta).$$

Obviously (2.3.10) holds for $\theta = \infty$.

The theorem now follows from (2.3.5) and (2.3.10).

REMARKS. (i) The above theorem generalises a result of Šeremeta [92], who obtained it under the following additional and restrictive conditions (I) and (II) :

(I) $\alpha(x)$ and $\beta(x)$ both belong to Λ_*.

(II) $\limsup\limits_{x \to \infty} \dfrac{d \log F(x,c)}{d \log x} < 1$ for every c, $0 < c < \sigma$,

where $\sigma = \infty$ if $\alpha \not\equiv \beta$ and $\sigma = 1$ if $\alpha \equiv \beta$.

Šeremeta's condition (II) requires that both $\alpha(x)$ and $\beta(x)$ are differentiable on $[a,\infty)$ while we have assumed only the continuity of $\alpha(x)$ and $\beta(x)$ on $[a,\infty)$.

It is further seen that the choices $\alpha(x) = \log_p x$, $p \ge 1$ and $\beta(x) = x^d$, $0 < d < \infty$ are permissible in our Theorem 2.3.1 but the corresponding theorem of Šeremeta is not applicable in these cases.

(ii) With the choices $\alpha(x) = \log_p x$, $p \ge 2$, and $\beta(x) = \log x$ or $\alpha(x) = \log_p x$, $p \ge 1$, and $\beta(x) = x^d$, $0 < d < \infty$, some results of Kapoor and Gopal [53 , 55] follow from the above theorem.

(iii) For the case $\alpha(x) = \beta(x) = \log x$, a coefficient characterization of (α,β)-order, analogous to that obtained in Theorem 2.3.1, of a function analytic in D_1 is due to Beuermann [13] and MacLane [65] and is given by (1.9.5) with $q = 2$.

Our next theorem gives a characterization of (α,β)-order $\rho(\alpha,\beta,f)$ of a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R <$ in terms of the ratio of the consecutive coefficients a_n's. The characterization holds for a subclass of functions analytic in D

THEOREM 2.3.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$ be of (α,β)-order $\rho(\alpha,\beta,f)$. Then*

(2.3.11) $$\rho(\alpha,\beta,f) \leq P(\theta_o)$$

where

(2.3.12) $$\theta_o = \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{\lambda_n - \lambda_{n-1}}{\log^+ |a_n/a_{n-1}|\, R^{\lambda_n-\lambda_{n-1}}}\right)} .$$

Further, if $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ is a nondecreasing function of n for $n > n_o$, then, in case $\alpha \not\equiv \beta$, equality holds in (2.3.11); for the case $\alpha \equiv \beta$ equality holds in (2.3.11) provided $\rho(\alpha,\alpha,f) \geq 1$.

PROOF. To prove (2.3.11) we first assume that $\rho(\alpha,\beta,f) > P(0)$. Then, by Lemma 2.2.2, we get

(2.3.13) $$\rho(\alpha,\beta,f) = \limsup_{r\to R} \frac{\alpha(\nu(r))}{\beta(R/(R-r))} .$$

Let the range set of $\nu(r)$ be $\{\lambda_{n_k}\}$ and $\tau(k)$ be the jump points of $\nu(r)$. Then $\tau(k) = |a_{n_{k-1}}/a_{n_k}|^{1/(\lambda_{n_k}-\lambda_{n_{k-1}})}$ is a strictly increasing sequence with $\tau(k) \to R$ as $k \to \infty$. Further, $\nu(r) = \lambda_{n_k}$ for $\tau(k) \leq r < \tau(k+1)$. Now, since $\beta(x) \in L^0$, by (2.3.13), we ha

$$(2.3.14) \quad \rho(\alpha,\beta,f) = \limsup_{k \to \infty} \frac{\alpha(\lambda_{n_k})}{\beta(\frac{R}{R-\tau(k)})}$$

$$= \limsup_{k \to \infty} \frac{\alpha(\lambda_{n_k})}{\beta(\frac{1}{\log (R/\tau(k))})}$$

$$= \limsup_{k \to \infty} \frac{\alpha(\lambda_{n_k})}{\beta\left(\dfrac{\lambda_{n_k}-\lambda_{n_{k-1}}}{\log^+ |a_{n_k}/a_{n_{k-1}}| R^{\lambda_{n_k}-\lambda_{n_{k-1}}}}\right)}$$

$$\leq \limsup_{k \to \infty} \frac{\alpha(\lambda_{n_k})}{\beta\left(\dfrac{\lambda_{n_k}-\lambda_{n_k-1}}{\log^+ |a_{n_k}/a_{n_k-1}| R^{\lambda_{n_k}-\lambda_{n_k-1}}}\right)}$$

$$\leq \limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{\lambda_n - \lambda_{n-1}}{\log^+ |a_n/a_{n-1}|\ R^{\lambda_n-\lambda_{n-1}}}\right)} .$$

This proves (2.3.11) for the case $\rho(\alpha,\beta,f) > P(0)$. For $\rho(\alpha,\beta,f) \leq P(0)$, (2.3.11) obviously holds.

Now, let $\psi(n)$ be a nondecreasing function of n for $n > n_o$ and let $\rho(\alpha,\beta,f) \geq P(0)$. To prove $\rho(\alpha,\beta,f) \geq P(\theta_o)$, we may assume without loss of generality that $\psi(n) \neq R$ for any finite value of n, since otherwise $\psi(n) = \Psi(n+1) = \ldots = R$ for all sufficiently large values of n and so $\theta_o = 0$ and $\rho(\alpha,\beta,f) \leq P(0$
As $\Psi(n)$ is nondecreasing for $n > n_o$, we have

$$\log |a_n| = \log |a_{n_o}| + \log |a_{n_o+1}/a_{n_o}| + \ldots + \log |a_n/a_{n-1}|$$

$$= \log |a_{n_o}| + (\lambda_{n_o+1} - \lambda_{n_o}) \log \frac{1}{\Psi(n_o)} + \ldots$$

$$\ldots + (\lambda_n - \lambda_{n-1}) \log \frac{1}{\Psi(n-1)}$$

$$\geq \log |a_{n_o}| + (\lambda_n - \lambda_{n_o}) \log \frac{1}{\Psi(n-1)}$$

and so

$$\log |a_n| R^{\lambda_n} \geq \log |a_{n_o}| R^{\lambda_{n_o}} + (\lambda_n - \lambda_{n_o}) \log \frac{R}{\Psi(n-1)}$$

$$= (1 + o(1)) \lambda_n \log \frac{R}{\Psi(n-1)}$$

or

$$\frac{1}{\log \frac{R}{\Psi(n-1)}} \geq \frac{(1+o(1)) \lambda_n}{\log |a_n| R^{\lambda_n}} . \tag{2.3.15}$$

Since $\beta(x) \in L^o$ is strictly increasing on $[a,\infty)$, for all sufficiently large values of n, we have

$$\beta\left(\frac{1}{\log \frac{R}{\Psi(n-1)}}\right) \geq \beta\left(\frac{(1+o(1))\lambda_n}{\log |a_n| R^{\lambda_n}}\right).$$

The above relation, since $\beta(x) \in L^o$, in view of Theorem 2.3.1, gives that

$$(2.3.16) \qquad \rho(\alpha,\beta,f) \geq \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{\lambda_n - \lambda_{n-1}}{\log |a_n/a_{n-1}|\, R^{\lambda_n-\lambda_{n-1}}}\right)} .$$

The theorem now follows from (2.3.14) and (2.3.16).

REMARKS. (i) For the case $\alpha(x) = \beta(x) = \log x$, a result analogous to Theorem 2.3.2 was obtained by Kapoor [50].

(ii) Let $\psi(n)$ be ultimately a nondecreasing function of n and let $n_k < n \leq n_{k+1}$, where $\{\lambda_{n_k}\}$ is the range set of $\nu(r)$. Then $\psi(n_k) = \psi(n_k+1) = \dots = \psi(n-1) = \tau(k+1)$, for large values of k. Thus

$$(2.3.17) \qquad \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\log \frac{R}{\psi(n-1)}}\right)} \leq \frac{\alpha(\lambda_{n_{k+1}})}{\beta\left(\dfrac{1}{\log \frac{R}{\tau(k+1)}}\right)} .$$

Taking the limit superior in (2.3.17) and using (2.3.14) we get (2.3.16). This gives an alternative proof of (2.3.16).

(iii) Equality in (2.3.11) need not hold if $\psi(n) = |a_n / a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ is ultimately not a nondecreasing function of n. This is shown in the following example.

EXAMPLE 2.3.1. Consider the function

$$F_N(z) = \sum_{n=2N}^{\infty} a_n z^n = f_1(z) + f_2(z)$$

$$= \sum_{n=N}^{\infty} (z/R)^{2n+1} + \sum_{n=N}^{\infty} \exp\left(\frac{2n}{\beta^{-1}(\frac{1}{d}\,\alpha(2n))}\right)(z/R)^{2n},$$

$$0 < R < \infty,$$

where $0 < d < \infty$ if $\alpha \not\equiv \beta$ and $1 < d < \infty$ if $\alpha \equiv \beta$, $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3) and N is so large that $\beta^{-1}(\frac{1}{d}\,\alpha(2N))$ is well defined. It is easily seen that $F_N(z)$ is analytic in D_R and $|a_n/a_{n+1}|$ is not a nondecreasing function of n. Further, by Theorem 2.3.1, it follows that the (α,β)-order of $F_N(z)$ is d. However, we have

$$\limsup_{n\to\infty} \frac{\alpha(2n)}{\beta\left(\dfrac{1}{\log^+|a_{2n}/a_{2n-1}|R}\right)} = \limsup_{n\to\infty} \frac{\alpha(2n)}{\beta\left(\dfrac{\beta^{-1}(\frac{1}{d}\alpha(2n))}{2n}\right)} = \infty,$$

since $\alpha^{-1}(\frac{1}{d}\,\alpha(x)) < x$ for $1 < d < \infty$ and if $\alpha \not\equiv \beta$, by (2.1.3), $\beta^{-1}(\frac{1}{d}\,\alpha(x))/x \to 0$ as $x \to \infty$, $0 < d < \infty$. Thus, for the function $F_N(z)$, strict inequality holds in (2.3.11).

2.4. We obtain the coefficient characterizations of lower (α,β)-order $\lambda(\alpha,\beta,f)$ of a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, in this section.

<u>THEOREM</u> 2.4.1. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>be of lower</u> (α,β)-<u>order</u> $\lambda(\alpha,\beta,f)$. <u>Then, if</u> $\alpha \not\equiv \beta$, <u>we have</u>

(2.4.1) $$\lambda(\alpha,\beta,f) \geq P(\,\theta(\{n_k\})),$$

<u>for any increasing sequence</u> $\{n_k\}$ <u>of positive integers, where</u>

(2.4.2) $$\theta(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\beta(\lambda_{n_k}/\log^+|a_{n_k}|\,R^{\lambda_{n_k}})}.$$

<u>If</u> $\alpha \equiv \beta$, <u>then equation</u> (2.4.1) <u>continues to hold provided</u> $\lambda(\alpha,\alpha,f) \geq 1$.

PROOF. First, let $P(0) < \theta(\{n_k\}) < \infty$ and $P(0) < \theta(\{n_k\}) - \varepsilon$, $\varepsilon > 0$. Then, there exists $k_o = k_o(\varepsilon)$ such that, for $k > k_o$, we have

$$\log |a_{n_k}| R^{\lambda_{n_k}} > \lambda_{n_k}/F(\lambda_{n_{k-1}}, 1/\bar{\theta}) \tag{2.4.3}$$

where $F(\lambda_{n_{k-1}}, 1/\bar{\theta}) = \beta^{-1}((1/\bar{\theta})\ \alpha(\lambda_{n_{k-1}}))$ and $\bar{\theta} = \theta(\{n_k\}) - \varepsilon$.
For $k > k_o$, set

$$r_k = R \exp\left(\frac{-1}{b\ F(\lambda_{n_{k-1}}, 1/\bar{\theta})}\right)$$

where b is a constant such that $1 < b < \infty$. Then, for $r_k \leq r \leq r_{k+1}$, using Cauchy's inequality and (2.4.3) we obtain

$$\begin{aligned}
\log M(r,f) &\geq \log |a_{n_k}| + \lambda_{n_k} \log r \\
&\geq \log |a_{n_k}| R^{\lambda_{n_k}} + \lambda_{n_k} \log (r_k/R) \\
&> \lambda_{n_k}/F(\lambda_{n_{k-1}}, 1/\bar{\theta}) + \lambda_{n_k} \log (r_k/R) \\
&= b\,\lambda_{n_k} \log (R/r_k) - \lambda_{n_k} \log (R/r_k) \\
&= (b-1) \log (R/r_k)\ \alpha^{-1}\left(\bar{\theta}\ \beta\left(\frac{1}{b \log (R/r_{k+1})}\right)\right) \\
&\geq (b-1) \log (R/r)\ \alpha^{-1}\left(\bar{\theta}\ \beta\left(\frac{1}{b \log (R/r)}\right)\right).
\end{aligned}$$

In view of the condition (2.1.3), the above relation gives that

$$\begin{aligned}
\alpha\left(\frac{b}{b-1} \log M(r,f)\right) &\geq \alpha\left(b \log (R/r)\ \alpha^{-1}\left(\bar{\theta}\ \beta\left(\frac{1}{b \log (R/r)}\right)\right)\right) \\
&\sim \bar{\theta}\ \beta\left(\frac{1}{b \log (R/r)}\right).
\end{aligned} \tag{2.4.4}$$

Since, $\beta(x) \in L^{O}$, we have

$$(2.4.5) \qquad \beta\left(\frac{R}{R-r}\right) \sim \beta\left(\frac{1}{\log (R/r)}\right)$$

as $r \to R$. From (2.4.4) and (2.4.5), since $\alpha(x) \in \Lambda$, we now get

$$\lambda(\alpha,\beta,f) \geq \bar{\theta} \liminf_{r \to R} \frac{\beta\left(\frac{1}{b \log (R/r)}\right)}{\beta\left(\frac{1}{\log (R/r)}\right)}.$$

Since $\beta(x) \in L^{O}$, applying Lemma 2.2.1, the above relation yields that $\lambda(\alpha,\beta,f) \geq \bar{\theta} = \theta(\{n_k\}) - \varepsilon$. Thus, letting $\varepsilon \to 0$, we get $\lambda(\alpha,\beta,f) \geq \theta(\{n_k\})$ and so

$$(2.4.6) \qquad \lambda(\alpha,\beta,f) \geq P(\theta(\{n_k\})).$$

Since, by the hypothesis of the theorem $\lambda(\alpha,\beta,f) \geq P(0)$, (2.4.6) obviously holds for $\theta(\{n_k\}) \leq P(0)$. For $\theta(\{n_k\}) = \infty$ the above analysis, with an arbitrarily large number in place of $\bar{\theta}$, gives that $\lambda(\alpha,\beta,f) = \infty$.

This proves the theorem.

COROLLARY. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$, *be of* (α,β)-*order* $\rho(\alpha,\beta,f)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$. *Further, assume that*

$$\text{(i)} \quad \lim_{n \to \infty} \alpha(\lambda_n)/\alpha(\lambda_{n+1}) = 1,$$

and

$$\text{(ii)} \quad \lim_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta(\lambda_n/\log^{+} |a_n| R^{\lambda_n})} = S \text{ exists with } 0 < 1/S < \sigma,$$

where $\sigma = \infty$ *if* $\alpha \not\equiv \beta$ *and* $\sigma = 1$ *if* $\alpha \equiv \beta$.

Then, $f(z)$ is of regular (α,β)-growth with $\rho(\alpha,\beta,f) = \lambda(\alpha,\beta,f) =$

REMARKS. (i) Taking $\alpha(x) = \log_p x$, $p \geq 2$, and $\beta(x) = \log x$ or $\alpha(x) = \log_p x$, $p \geq 1$, and $\beta(x) = x^d$, $0 < d < \infty$, some results of Kapoor and Gopal ([53], [55]) follow from Theorem 2.4.1.

(ii) A coefficient formula, analogous to that of Theorem 2.3.1, does not hold, in general, for the lower (α,β)-order $\lambda(\alpha,\beta,f)$. For, consider the function $F_N(z) = f_1(z) + f_2(z)$, in Example 2.3.1. Then, by the corollary of Theorem 2.4.1, $f_2(z)$ is of regular (α,β)-growth with (α,β)-order $\rho(\alpha,\beta,f_2) = d$. Furth for all r, sufficiently near to R, we have

$$M(r,f_2) \leq M(r,F_N) = M(r,f_1) + M(r,f_2) \leq 2M(r,f_2).$$

The above relation shows that the function $F_N(z)$ is also of regular (α,β)-growth with lower (α,β)-order $\lambda(\alpha,\beta,F_N) = d$. But

$$\liminf_{n \to \infty} \frac{\alpha(n)}{\beta(n/\log^+ |a_n| \, R^n)} = 0.$$

(ii) Following are some examples of functions, analytic in D_R, which are of fast growth (Section 1.8) and are such that the growth parameters of Kapoor and Gopal ([53], [55]) fail to give any specific information about their growth. However, these functions have nonzero finite growth parameters in our sense. Thus, the growth of these functions can not be studied precisely by confining to the growth parameters of Kapoor and Gopal, while the growth of these functions can be measured precisely by using our growth parameters.

EXAMPLE. Let

$$F_1(z) = \sum_{n=1}^{\infty} a_{n,1}(z/R)^n, \; 0 < R < \infty,$$

where

$$a_{n,1} = \exp\,(n/\exp\,(\log n)^{1/2}).$$

Then $F_1(z)$ is analytic in D_R. Applying (1.9.4) to $F_1(z)$ we see that the order $\rho(F_1)$ of $F_1(z)$ satisfies

$$\rho(F_1) = \infty,$$

while, by Theorem 2.3.1, we have

$$\rho(\log_p, \log, F_1) = 0, \; p \geq 2.$$

On the other hand, from the corollary of Theorem 2.4.1, we have

$$\rho(\log_2, \log_2, F_1) = \lambda(\log_2, \log_2, F_1) = 2.$$

EXAMPLE. Let

$$F_2(z) = \sum_{n=9}^{\infty} a_{n,2}(z/R)^n, \; 0 < R < \infty,$$

$$F_3(z) = \sum_{n=9}^{\infty} a_{n,3}(z/R)^n, \; 0 < R < \infty,$$

where

$$a_{n,2} = \exp\,(n/\exp\,\exp\,(2 \log_3 n))$$

and

$$a_{n,3} = \exp\,(n/\exp\,(\log_3 n)^{1/d}), \; 0 < d < 1.$$

Clearly, both $F_2(z)$ and $F_3(z)$ are analytic in D_R. The order $\rho(F_2)$ of $F_2(z)$ and the order $\rho(F_3)$ of $F_3(z)$ satisfy $\rho(F_2)=\rho(F_3)=\infty$. Further, $\rho(\log_p, \log, F_2) = 0$ for $p \geq 2$, $\rho(\log_2, \log, F_3) = \infty$ and $\rho(\log_p, \log, F_3) = 0$ for $p \geq 3$. However, $\rho(\log_3, \log_2, F_2) = 1/2$

and $\rho(\alpha_o,\beta_o,F_3) = \lambda(\alpha_o,\beta_o,F_3) = 1$ for $\alpha_o(x) = \log_2 x$ and $\beta_o(x) = \exp(\log x)^d$.

THEOREM 2.4.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$, *be of lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$. *Then, if* $\alpha \not\equiv \beta$, *we have*

$$(2.4.7) \qquad \lambda(\alpha,\beta,f) \geq P(\phi(\{n_k\}))$$

where

$$(2.4.8) \qquad \phi(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\beta\left(\dfrac{\lambda_{n_k} - \lambda_{n_{k-1}}}{\log^+ |a_{n_k}/a_{n_{k-1}}| R^{\lambda_{n_k}-\lambda_{n_{k-1}}}}\right)},$$

for any increasing sequence $\{n_k\}$ *of positive integers. The equation* (2.4.7) *continues to hold for* $\alpha \equiv \beta$, *provided* $\lambda(\alpha,\alpha,f) \geq 1$

PROOF. First, let $P(0) < \phi(\{n_k\}) < \infty$ and $P(0) < \phi(\{n_k\}) - \varepsilon$, $\varepsilon > 0$. Then, there exists $k_o = k_o(\varepsilon)$ such that, for all $k > k_o$, by (2.4.8), we have

$$(2.4.9) \quad \log|a_{n_k}/a_{n_{k-1}}| R^{\lambda_{n_k}-\lambda_{n_{k-1}}} > (\lambda_{n_k}-\lambda_{n_{k-1}})/F(\lambda_{n_{k-1}},1/\bar{\phi})$$

where $F(\lambda_{n_{k-1}},1/\bar{\phi}) = \beta^{-1}((1/\bar{\phi})\,\alpha(\lambda_{n_{k-1}}))$ and $\bar{\phi} = \phi(\{n_k\}) - \varepsilon$. Now

$$|a_{n_k}|R^{\lambda_{n_k}} = |a_{n_{k_o}}|R^{\lambda_{n_{k_o}}} \prod_{m=k_o+1}^{k} |a_{n_m}/a_{n_{m-1}}|R^{\lambda_{n_m}-\lambda_{n_{m-1}}}.$$

and so, by (2.4.9), we get

$$(2.4.10) \quad \log|a_{n_k}|R^{\lambda_{n_k}} > \log|a_{n_{k_o}}|R^{\lambda_{n_{k_o}}} + \sum_{m=k_o+1}^{k} (\lambda_{n_m}-\lambda_{n_{m-1}})/F(\lambda_{n_{m-1}},1/\bar{\phi})$$

$$> \log|a_{n_{k_o}}|R^{\lambda_{n_{k_o}}} + (\lambda_{n_k}-\lambda_{n_{k_o}})/F(\lambda_{n_{k-1}},1/\bar{\phi}).$$

From (2.4.10), on using Cauchy's inequality, we obtain

$$\log M(r,f) \geq \log |a_{n_k}| + \lambda_{n_k} \log r$$

$$= \log |a_{n_k}| R^{\lambda_{n_k}} - \lambda_{n_k} \log (R/r)$$

$$\geq \log |a_{n_{k_o}}| R^{\lambda_{n_{k_o}}} + (\lambda_{n_k} - \lambda_{n_{k_o}})/F(\lambda_{n_{k-1}}, 1/\bar{\phi})$$

$$- \lambda_{n_k} \log (R/r)$$

or

(2.4.11) $(1+o(1)) \log M(r,f) \geq \lambda_{n_k}/F(\lambda_{n_{k-1}}, 1/\bar{\phi}) - \lambda_{n_k} \log (R/r)$.

For $k > k_o$, we set

$$r_k = R \exp \left(\frac{-1}{b\ F(\lambda_{n_{k-1}}, 1/\bar{\phi})}\right)$$

where b is a constant such that $1 < b < \infty$. Then, for $r_k \leq r \leq r_{k+1}$, (2.4.11) gives

$$(1+o(1)) \log M(r,f) \geq b\ \lambda_{n_k} \log (R/r_k) - \lambda_{n_k} \log (R/r_k)$$

$$= (b-1) \log (R/r_k) \alpha^{-1}\left(\bar{\phi}\ \beta\left(\frac{1}{b \log (R/r_{k+1})}\right)\right)$$

$$\geq (b-1) \log(R/r) \alpha^{-1}\left(\bar{\phi}\ \beta\left(\frac{1}{b \log (R/r)}\right)\right).$$

The above relation, in view of (2.1.3), gives that

$$\alpha\left(\frac{b}{b-1}(1+o(1)) \log M(r,f)\right) \geq \alpha\left(b \log(R/r) \alpha^{-1}\left(\bar{\phi}\ \beta\left(\frac{1}{b \log (R/r)}\right)\right)\right)$$

$$\sim \bar{\phi}\ \beta\left(\frac{1}{b \log (R/r)}\right).$$

From (2.4.5) and the above relation, since $\alpha(x) \in \Lambda$, we have

$$\lambda(\alpha,\beta,f) \geq \bar{\Phi} \liminf_{r \to R} \frac{\beta\left(\frac{1}{b \log (R/r)}\right)}{\beta\left(\frac{1}{\log (R/r)}\right)} .$$

And so, since $\beta(x) \in L^{o}$, using Lemma 2.2.1, we get $\lambda(\alpha,\beta,f) \geq \phi(\{n_k\}) - \varepsilon$. Letting $\varepsilon \to 0$, we now get $\lambda(\alpha,\beta,f) \geq \phi(\{n_k\})$ and thus

(2.4.12) $$\lambda(\alpha,\beta,f) \geq P(\phi(\{n_k\})).$$

Obviously, (2.4.12) holds for $\phi(\{n_k\}) \leq P(0)$, since, by the hypothesis of the theorem $\lambda(\alpha,\beta,f) \geq P(0)$. For $\phi(\{n_k\}) = \infty$, the above arguments, with an arbitrarily large number in place of $\bar{\Phi}$, give that $\lambda(\alpha,\beta,f) = \infty$. This proves the theorem.

THEOREM 2.4.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$, *be of lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$. *Assume that* $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *is a nondecreasing function of* n *for* $n > n_o$. *Then, if* $\alpha \not\equiv \beta$, *we have*

(2.4.13) $$\lambda(\alpha,\beta,f) = P(\theta_*)$$

where

$$\theta_* = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\beta(\lambda_n/\log^+ |a_n| R^{\lambda_n})} .$$

The equation (2.4.13) *continues to hold for* $\alpha \equiv \beta$ *if* $\lambda(\alpha,\alpha,f) \geq 1$.

PROOF. We can assume without loss of generality that $\psi(n) > \psi(n-1)$ for infinitely many values of n. Since, otherwise $\psi(n) = \psi(n+1) = \ldots = R$ for all sufficiently large values of n

and so $\{|a_n|R^{\lambda_n}\}$ is bounded, which gives that $\theta_* \leq P(0)$ and $\lambda(\alpha,\beta,f) \leq P(0)$.

Clearly $\Psi(n) \to R$ as $n \to \infty$. When $\Psi(n) > \Psi(n-1)$, the term $a_n z^{\lambda_n}$ becomes the maximum term and we have

$$\mu(r) = |a_n| r^{\lambda_n},\ \nu(r) = \lambda_n \text{ for } \Psi(n-1) \leq r < \Psi(n).$$

Now, by Lemma 2.2.2, we have

$$(2.4.14) \quad \lambda(\alpha,\beta,f) = P(\theta_2),\ \theta_2 = \liminf_{r \to R} \frac{\alpha(\nu(r))}{\beta(R/(R-r))}.$$

Let $P(0) < \lambda(\alpha,\beta,f) < \infty$ and let $\varepsilon > 0$ be such that $P(0) < \lambda(\alpha,\beta,f)-\varepsilon$ Then, for $r > r_o = r_o(\varepsilon)$, by (2.4.14), we get

$$\nu(r) > \alpha^{-1}(\bar{\lambda}\ \beta(R/(R-r)))$$

where $\bar{\lambda} = \lambda(\alpha,\beta,f) - \varepsilon$. Let $a_{n_1} z^{\lambda_{n_1}}$ and $a_{n_2} z^{\lambda_{n_2}}$ ($n_1 > n_o$, $\psi(n_1-1) > r_o$) be two consecutive maximum terms so that $n_1 \leq n_2-1$. Suppose that $n_1 < n \leq n_2$. Since $a_{n_1} z^{\lambda_{n_1}}$ is maximum term we have $\nu(r) = \lambda_{n_1}$ for $\psi(n_1-1) \leq r < \psi(n_1)$. Hence, for every r in this interval we obtain

$$\lambda_{n_1} > \alpha^{-1}(\bar{\lambda}\ \beta(R/(R-r))).$$

Thus, choosing $r = \psi(n_1) - b(R-\psi(n_1))$, $0 < b < 1$, we get

$$\lambda_{n_1} > \alpha^{-1}(\bar{\lambda}\ \beta(\frac{R}{(1+b)(R-\psi(n_1))}))$$

since $\psi(n_1) - b(R-\psi(n_1)) < \psi(n_1)$. Further we have $\psi(n_1) = \psi(n_1+1) = \dots = \psi(n-1)$. Hence

$$(2.4.15) \qquad \lambda_{n-1} \geq \lambda_{n_1} \geq \alpha^{-1}\left(\bar{\lambda}\ \beta\left(\frac{R}{(1+b)(R-\Psi(n-1))}\right)\right)$$

$$\geq \alpha^{-1}\left(\bar{\lambda}\ \beta\left(\frac{1}{(1+b)\log(R/\psi(n-1))}\right)\right).$$

Now, since $\psi(n)$ is a nondecreasing function of n for $n > n_o$, by (2.3.15), we have

$$(2.4.16) \qquad \frac{1}{\log(R/\Psi(n-1))} \geq (1+o(1))\frac{\lambda_n}{\log|a_n|\ R^{\lambda_n}}.$$

From (2.4.15) and (2.4.16), we get

$$\frac{\alpha(\lambda_{n-1})}{\beta(\lambda_n/\log|a_n|R^{\lambda_n})} \geq \bar{\lambda}\ \frac{\beta(((1+o(1))/(1+b))\ \lambda_n/\log|a_n|\ R^{\lambda_n})}{\beta(\lambda_n/\log|a_n|\ R^{\lambda_n})}.$$

Since $\beta(x) \in L^o$, applying Lemma 2.2.1, we get $\theta_* \geq \bar{\lambda} = \lambda(\alpha,\beta,f)-\varepsilon$. Letting $\varepsilon \to 0$, we get $\theta_* \geq \lambda(\alpha,\beta,f)$ and so, since $\lambda(\alpha,\beta,f) > P(0)$,

$$(2.4.17) \qquad P(\theta_*) \geq \lambda(\alpha,\beta,f).$$

Obviously (2.4.17) holds for $\lambda(\alpha,\beta,f) = P(0)$. For $\lambda(\alpha,\beta,f) = \infty$, the above arguments with an arbitrarily large number in place of $\bar{\lambda}$ give that $\theta_* = \infty$.

In view of (2.4.17) and Theorem 2.4.1 the proof of the theorem is complete.

<u>REMARK</u>. A result analogous to Theorem 2.4.3, has been obtained by Kapoor [50, 51] for the case $\alpha(x) = \beta(x) = \log x$.

<u>THEOREM</u> 2.4.4. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>be of lower</u> (α,β)-<u>order</u> $\lambda(\alpha,\beta,f)$. <u>Assume that</u> $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ <u>is a nondecreasing function of</u> n <u>for</u> $n > n_o$.

Then, if $\alpha \not\equiv \beta$, we have

(2.4.18) $$\lambda(\alpha;\beta,f) = P(\phi_*)$$

where

$$\phi_* = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\beta\left(\dfrac{\lambda_n - \lambda_{n-1}}{\log^+ |a_n/a_{n-1}|\ R^{\lambda_n - \lambda_{n-1}}}\right)} .$$

The equation (2.4.18) continues to hold for $\alpha \equiv \beta$ provided $\lambda(\alpha,\alpha,f) \geq 1$.

PROOF. First, let $P(0) < \lambda(\alpha,\beta,f) < \infty$. Then $\psi(n) > \psi(n-1)$ for infinitely many values of n, since otherwise, by the proof of Theorem 2.4.3, $\lambda(\alpha,\beta,f) \leq P(0)$. Further, $\psi(n) \to R$ as $n \to \infty$. When $\psi(n) > \psi(n-1)$ the term $a_n z^{\lambda_n}$ becomes the maximum term and we have

$$\mu(r) = |a_n| r^{\lambda_n} \text{ for } \psi(n-1) \leq r < \psi(n).$$

Now, by Lemma 2.2.2, it follows that

(2.4.19) $$\lambda(\alpha,\beta,f) = \liminf_{r \to R} \frac{\alpha(\log \mu(r))}{\beta(R/(R-r))} .$$

Thus, given $\varepsilon > 0$, $\lambda(\alpha,\beta,f) - \varepsilon > P(0)$, there exists $r_o = r_o(\varepsilon)$ such that for $r > r_o$ we obtain

$$\alpha(\log \mu(r)) > \bar{\lambda}\ \beta(R/(R-r))$$

where $\bar{\lambda} = \lambda(\alpha,\beta,f)-\varepsilon$. Let $a_{n_1} z^{\lambda_{n_1}}$ and $a_{n_2} z^{\lambda_{n_2}}$ $(n_1 > n_o, \psi(n_1-1) > r_o)$ be two consecutive maximum terms so that $n_1 \leq n_2-1$. Then

$$\alpha(\log |a_{n_2}| + \lambda_{n_2} \log r) > \bar{\lambda}\ \beta(R/(R-r))$$

for all r satisfying $\psi(n_2-1) \leq r < \psi(n_2)$. Let $n_1 \leq n \leq n_2-1$. It is easily seen that $\psi(n_1) = \psi(n_1+1) = \ldots = \psi(n) = \ldots$
$\ldots = \psi(n_2-1)$ and that

$$|a_n|\, r^{\lambda_n} = |a_{n_2}|\, r^{\lambda_{n_2}} \text{ for } r = \psi(n).$$

Hence

(2.4.20) $\qquad \alpha(\log |a_n| + \lambda_n \log \psi(n)) > \bar{\lambda}\ \beta(R/(R-\psi(n)))$.

Since $\Psi(n)$ is nondecreasing for $n > n_o$ we have
$\log (\Psi(n)\, |a_n|^{1/\lambda_n}) < 1$ for all sufficiently large values of n. Thus, for all sufficiently large values of n, by (2.4.20), we have

$$\begin{aligned} \alpha(\lambda_n) &> \bar{\lambda}\ \beta(R/(R-\psi(n))) \\ &\sim \bar{\lambda}\ \beta(1/\log (R/\Psi(n))) \\ &= \bar{\lambda}\ \beta\left(\frac{\lambda_{n+1} - \lambda_n}{\log |a_{n+1}/a_n|\ R^{\lambda_{n+1}-\lambda_n}}\right), \end{aligned}$$

since $\beta(x) \in L^o$ and $R/(R-\psi(n)) \sim 1/\log (R/\Psi(n))$ as $n \to \infty$. The above relation on passing to limits gives that $\phi_* \geq \bar{\lambda} = \lambda(\alpha,\beta,f)-\varepsilon$. Letting $\varepsilon \to 0$, we get $\phi_* \geq \lambda(\alpha,\beta,f)$ and thus

(2.4.21) $\qquad P(\phi_*) \geq \lambda(\alpha,\beta,f)$.

Obviously (2.4.21) holds for $\lambda(\alpha,\beta,f) = P(0)$. For $\lambda(\alpha,\beta,f) = \infty$ the above arguments with an arbitrarily large number in place of $\bar{\lambda}$ give that $\phi_* = \infty$.

On combining (2.4.21) with Theorem 2.4.2 we get the theorem.

for all r satisfying $\psi(n_2-1) \leq r < \psi(n_2)$. Let $n_1 \leq n \leq n_2-1$. It is easily seen that $\psi(n_1) = \psi(n_1+1) = \ldots = \psi(n) = \ldots$ $\ldots = \psi(n_2-1)$ and that

$$|a_n| \, r^{\lambda_n} = |a_{n_2}| \, r^{\lambda_{n_2}} \text{ for } r = \psi(n).$$

Hence

$$\alpha(\log |a_n| + \lambda_n \log \psi(n)) > \bar{\lambda}\, \beta(R/(R-\psi(n))). \tag{2.4.20}$$

Since $\Psi(n)$ is nondecreasing for $n > n_o$ we have $\log (\Psi(n) \, |a_n|^{1/\lambda_n}) < 1$ for all sufficiently large values of n. Thus, for all sufficiently large values of n, by (2.4.20), we have

$$\begin{aligned} \alpha(\lambda_n) &> \bar{\lambda}\, \beta(R/(R-\psi(n))) \\ &\sim \bar{\lambda}\, \beta(1/\log (R/\Psi(n))) \\ &= \bar{\lambda}\, \beta\left(\frac{\lambda_{n+1} - \lambda_n}{\log |a_{n+1}/a_n| \, R^{\lambda_{n+1}-\lambda_n}}\right), \end{aligned}$$

since $\beta(x) \varepsilon L^o$ and $R/(R-\psi(n)) \sim 1/\log (R/\Psi(n))$ as $n \to \infty$. The above relation on passing to limits gives that $\phi_* \geq \bar{\lambda} = \lambda(\alpha,\beta,f)-$. Letting $\varepsilon \to 0$, we get $\phi_* \geq \lambda(\alpha,\beta,f)$ and thus

$$P(\phi_*) \geq \lambda(\alpha,\beta,f). \tag{2.4.21}$$

Obviously (2.4.21) holds for $\lambda(\alpha,\beta,f) = P(O)$. For $\lambda(\alpha,\beta,f) = \infty$ the above arguments with an arbitrarily large number in place of $\bar{\lambda}$ give that $\phi_* = \infty$.

On combining (2.4.21) with Theorem 2.4.2 we get the theorem.

and

$$\bar{n}_k = [\alpha^{-1}(2\alpha(n_k))] + 1$$

where $[x]$ denotes the integral part of x. Let m be a positive integer such that $m \geq n_1$. We define

$$r_m = \exp\left(\frac{1}{\beta^{-1}(\alpha(m)/2)}\right) \quad \text{for } n_k \leq m < \bar{n}_k$$

and if $\bar{n}_k < n_{k+1}$, we define

$$r_m = \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1})/2)}\right) + \frac{\exp\left(\frac{1}{\beta^{-1}(\alpha(m))}\right) - \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1}))}\right)}{\beta^{-1}(\alpha(n_{k+1}))}$$

$$\text{for } \bar{n}_k \leq m < n_{k+1}.$$

Now, let $r_1, r_2, \ldots, r_{n_1-1}$ be each chosen equal to 1 and let

$$f_o(z) = \sum_{k=1}^{\infty} (r_1 r_2 \cdots r_k)\, z^k.$$

Since $r_k \to 1$ as $k \to \infty$, it follows that $f_o(z)$ is analytic in $|z| < 1$.

In order to apply Theorems 2.3.2 and 2.4.4 to the function $f_o(z)$ we prove that r_m is ultimately a nonincreasing function of m. For this, it is sufficient to show that, for all sufficientl large values of k, we have $r_{\bar{n}_k-1} \geq r_{\bar{n}_k}$, which is equivalent to

$$\text{(2.4.22)} \quad \exp\left(\frac{1}{\beta^{-1}(\alpha(\bar{n}_k-1)/2)}\right) \geq \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1})/2)}\right)$$

$$+ \frac{\exp\left(\frac{1}{\beta^{-1}(\alpha(\bar{n}_k))}\right) - \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1}))}\right)}{\beta^{-1}(\alpha(n_{k+1}))}$$

<u>REMARKS</u>. (i) A coefficient characterization of lower q-order, $q \geq 3$, due to Kapoor and Gopal [53], follows from Theorem 2.4.3 on taking $\alpha(x) = \log_p x$, $p \geq 2$, and $\beta(x) = \log x$; while with the same choice of $\alpha(x)$ and $\beta(x)$, Theorem 2.4.4 gives a new characterization of lower q-order, for $q \geq 3$.

(ii) With $\alpha(x) = \log_p x$, $p \geq 1$, $\beta(x) = x^d$, $0 < d < \infty$, Theorem 2.4.3 improves a characterization of lower q-type, for $q \geq 3$, in (1.9.9), due to Kapoor and Gopal [53, 55], which was obtained under the additional restriction that $\log_p \lambda_n \sim \log_p \lambda_{n+1}$ as $n \to \infty$. With the same choices of $\alpha(x)$ and $\beta(x)$, Theorem 2.4.4 gives a new characterization of lower q-type, for $q \geq 3$.

(iii) For the case $\alpha(x) = \beta(x) = \log x$, a result analogous to Theorem 2.4.4, has been obtained by Kapoor [50] and Kapoor and Juneja [56].

(iv) The condition that $\Psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ is ultimately a nondecreasing function of n does not imply that the function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, is of regular (α,β)-growth. This is shown in the following example.

<u>EXAMPLE</u>. Let $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, $\alpha \not\equiv \beta$, and let $\alpha(x)$, $\beta(x)$ satisfy (2.1.3). We choose a positive integer n_1 such that $\alpha(n_1)$ is well defined. For an integer $k \geq 1$, let

$$n_{k+1} = [\alpha^{-1}(4\alpha(n_k))] + 1$$

and

$$\bar{n}_k = [\alpha^{-1}(2\alpha(n_k))] + 1$$

where $[x]$ denotes the integral part of x. Let m be a positive integer such that $m \geq n_1$. We define

$$r_m = \exp\left(\frac{1}{\beta^{-1}(\alpha(m)/2)}\right) \quad \text{for } n_k \leq m < \bar{n}_k$$

and if $\bar{n}_k < n_{k+1}$, we define

$$r_m = \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1})/2)}\right) + \frac{\exp\left(\frac{1}{\beta^{-1}(\alpha(m))}\right) - \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1}))}\right)}{\beta^{-1}(\alpha(n_{k+1}))}$$

$$\text{for } \bar{n}_k \leq m < n_{k+1}.$$

Now, let $r_1, r_2, \ldots, r_{n_1 - 1}$ be each chosen equal to 1 and let

$$f_o(z) = \sum_{k=1}^{\infty} (r_1 r_2 \cdots r_k)\, z^k.$$

Since $r_k \to 1$ as $k \to \infty$, it follows that $f_o(z)$ is analytic in $|z| < 1$.

In order to apply Theorems 2.3.2 and 2.4.4 to the function $f_o(z)$ we prove that r_m is ultimately a nonincreasing function of m. For this, it is sufficient to show that, for all sufficiently large values of k, we have $r_{\bar{n}_k - 1} \geq r_{\bar{n}_k}$, which is equivalent to

$$\exp\left(\frac{1}{\beta^{-1}(\alpha(\bar{n}_k - 1)/2)}\right) \geq \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1})/2)}\right) + \frac{\exp\left(\frac{1}{\beta^{-1}(\alpha(\bar{n}_k))}\right) - \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1}))}\right)}{\beta^{-1}(\alpha(n_{k+1}))}. \tag{2.4.22}$$

Now, since $\bar{n}_k - 1 \leq \alpha^{-1}(2\alpha(n_k))$, we get $\beta^{-1}(\alpha(\bar{n}_k-1)/2) \leq \beta^{-1}(\alpha(n_k$ and so

$$(2.4.23) \qquad \exp\left(\frac{1}{\beta^{-1}(\alpha(\bar{n}_k-1)/2)}\right) \geq \exp\left(\frac{1}{\beta^{-1}(\alpha(n_k))}\right).$$

Further, since $n_{k+1} > \alpha^{-1}(4\alpha(n_k))$, we get $\alpha(n_{k+1}) > 4\alpha(n_k)$.

Hence $\beta^{-1}(\alpha(n_{k+1})/2) > \beta^{-1}(2\alpha(n_k))$ and $\beta^{-1}(\alpha(n_{k+1})) > \beta^{-1}(4\alpha(n_k)$

From the relation $\bar{n}_k > \alpha^{-1}(2\alpha(n_k))$ we get $\beta^{-1}(\alpha(\bar{n}_k)) > \beta^{-1}(2\alpha(n_k$

Thus

$$(2.4.24) \quad \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1})/2)}\right) + \frac{\exp\left(\frac{1}{\beta^{-1}(\alpha(\bar{n}_k))}\right) - \exp\left(\frac{1}{\beta^{-1}(\alpha(n_{k+1}))}\right)}{\beta^{-1}(\alpha(n_{k+1}))}$$

$$\leq \exp\left(\frac{1}{\beta^{-1}(2\alpha(n_k))}\right) + \frac{\exp\left(\frac{1}{\beta^{-1}(2\alpha(n_k))}\right) - \exp(0)}{\beta^{-1}(4\alpha(n_k))}.$$

In view of (2.4.23) and (2.4.24), to establish (2.4.22), it is sufficient to show that, for all sufficiently large values of k, we have

$$(2.4.25) \quad \exp\left(\frac{1}{\beta^{-1}(x_k)}\right) - \exp\left(\frac{1}{\beta^{-1}(2x_k)}\right) \geq \frac{\exp\left(\frac{1}{\beta^{-1}(2x_k)}\right) - \exp(0)}{\beta^{-1}(4x_k)},$$

where $x_k = \alpha(n_k)$.

By mean value theorem, we have

$$\exp\left(\frac{1}{\beta^{-1}(x_k)}\right) - \exp\left(\frac{1}{\beta^{-1}(2x_k)}\right) = \left(\frac{1}{\beta^{-1}(x_k)} - \frac{1}{\beta^{-1}(2x_k)}\right) \exp(t_k),$$

where $1/\beta^{-1}(2x_k) < t_k < 1/\beta^{-1}(x_k)$. Since $\exp x$ is an increasing function of x, by the above identity, we have

$$(2.4.26)\qquad \exp\left(\frac{1}{\beta^{-1}(x_k)}\right) - \exp\left(\frac{1}{\beta^{-1}(2x_k)}\right)$$
$$\geq \left(\frac{1}{\beta^{-1}(x_k)} - \frac{1}{\beta^{-1}(2x_k)}\right) \exp\left(\frac{1}{\beta^{-1}(2x_k)}\right).$$

Similarly, it can be shown that

$$(2.4.27)\qquad \exp\left(\frac{1}{\beta^{-1}(2x_k)}\right) - \exp(0) \leq \frac{1}{\beta^{-1}(2x_k)} \exp\left(\frac{1}{\beta^{-1}(2x_k)}\right).$$

By (2.4.26) and (2.4.27), to prove (2.4.25), it is sufficient to show that

$$\frac{1}{\beta^{-1}(x_k)} - \frac{1}{\beta^{-1}(2x_k)} \geq \frac{1}{\beta^{-1}(2x_k)\ \beta^{-1}(4x_k)}$$

or, equivalently, to show that

$$(2.4.28)\qquad \frac{\beta^{-1}(2x_k)}{\beta^{-1}(x_k)} - 1 \geq \frac{1}{\beta^{-1}(4x_k)},$$

for all sufficiently large values of k.

It is easily seen that

$$(2.4.29)\qquad \liminf_{x \to \infty} \frac{\beta^{-1}(2x)}{\beta^{-1}(x)} = K > 1.$$

For, suppose on the contrary

$$\liminf_{x \to \infty} \frac{\beta^{-1}(2x)}{\beta^{-1}(x)} = 1.$$

Then,

$$\beta^{-1}(2x_k^*) = (1+o(1))\ \beta^{-1}(x_k^*)$$

for a sequence of values $\{x_k^*\}$, $x_k^* \to \infty$ as $k \to \infty$. Since $\beta(x) \in L^O$, applying $\beta(x)$ to both the sides of the above relation, we get

$$2x_k^* = \beta((1+o(1))\ \beta^{-1}(x_k^*)) \sim x_k^*$$

which is false and so (2.4.29) is true.

Now, since $\beta^{-1}(4x_k) \to \infty$ as $k \to \infty$, it follows from (2.4.29) that (2.4.28) holds for all sufficiently large values of k. Hence r_m is ultimately a nonincreasing function of m.

Applying Theorem 2.3.2 to $f_o(z)$, we now have

$$(2.4.30)\quad \rho(\alpha,\beta,f_o) = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(\frac{1}{\log r_n})} \geq \limsup_{k\to\infty} \frac{\alpha(n_k)}{\beta(\frac{1}{\log r_{n_k}})}$$

$$= \limsup_{k\to\infty} \frac{\alpha(n_k)}{\beta(\beta^{-1}(\alpha(n_k)/2))} = 2.$$

On the other hand, since $\beta(x) \in L^o$ and $\log r_n \sim \frac{r_n - 1}{r_n}$, as $n \to \infty$, applying Theorem 2.4.4 to $f_o(z)$, we get

$$\lambda(\alpha,\beta,f_o) = \liminf_{n\to\infty} \frac{\alpha(n-1)}{\beta(\frac{1}{\log r_n})} = \liminf_{n\to\infty} \frac{\alpha(n-1)}{\beta(\frac{1}{r_n - 1})}.$$

The above relation gives

$$(2.4.31)\qquad \lambda(\alpha,\beta,f_o) \leq \liminf_{k\to\infty} \frac{\alpha(\bar{n}_k - 1)}{\beta(\frac{1}{r_{\bar{n}_k} - 1})}.$$

To estimate the right hand side of (2.4.31), by (2.4.24), we have

$$r_{\bar{n}_k} \leq \exp\left(\frac{1}{\beta^{-1}(2\alpha(n_k))}\right) + \frac{\exp\left(\frac{1}{\beta^{-1}(2\alpha(n_k))}\right) - 1}{\beta^{-1}(4\alpha(n_k))}$$

and so, since $r - 1 \leq r \log r$, $r > 1$, (2.4.31) gives

$$\lambda(\alpha,\beta,f_o) \le \liminf_{k\to\infty} \frac{\alpha(\bar{n}_k-1)}{\beta\left(\dfrac{1}{(1+s_k)\left(\exp\left(\dfrac{1}{\beta^{-1}(2\alpha(n_k))}\right)-1\right)}\right)}$$

$$\le \liminf_{k\to\infty} \frac{\alpha(\bar{n}_k-1)}{\beta\left(\dfrac{\beta^{-1}(2\alpha(n_k))}{(1+s_k)\ \exp\left(\dfrac{1}{\beta^{-1}(2\alpha(n_k))}\right)}\right)},$$

where $s_k = 1/\beta^{-1}(4\alpha(n_k)) \to 0$ as $k \to \infty$. Since $\beta(x) \in L^O$, the above inequality now gives

(2.4.32) $$\lambda(\alpha,\beta,f_o) \le \liminf_{k\to\infty} \frac{\alpha(\bar{n}_k-1)}{2\alpha(n_k)} \le 1.$$

By (2.4.30) and (2.4.32) it follows that $f_o(z)$ is of irregular (α,β)-growth.

We now obtain coefficient characterizations of lower (α,β)-order of a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, that do not require the nondecreasing nature of the function $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$.

<u>THEOREM</u> 2.4.5. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic</u> <u>in</u> D_R, $0 < R < \infty$ <u>be</u> <u>of</u> <u>lower</u> (α,β)-<u>order</u> $\lambda(\alpha,\beta,f)$. <u>Then</u>, <u>if</u> $\alpha \not\equiv \beta$, <u>we</u> <u>have</u>

(2.4.33) $$\lambda(\alpha,\beta,f) = \max_{\{n_k\}} \left[P(\theta(\{n_k\})) \right]$$

<u>and</u>

(2.4.34) $$\lambda(\alpha,\beta,f) = \max_{\{n_k\}} \left[P(\phi(\{n_k\})) \right]$$

where $\theta(\{n_k\})$ and $\phi(\{n_k\})$ are defined by (2.4.2) and (2.4.8), respectively, and the maximum in (2.4.33) and (2.4.34) is taken over all increasing sequences $\{n_k\}$ of positive integers. The coefficient characterizations (2.4.33) and (2.4.34) continue to hold for $\alpha \equiv \beta$ provided $\lambda(\alpha,\alpha,f) \geq 1$.

PROOF. By Theorems 2.4.1 and 2.4.2 the assertion of the theorem obviously holds if $\lambda(\alpha,\beta,f) = P(0)$. Thus, let $\lambda(\alpha,\beta,f) > P(0)$. Now, consider the function $g(z) = \sum_{m=0}^{\infty} a_{n_m} z^{\lambda_{n_m}}$, where $\{\lambda_{n_m}\}$ is the sequence of the principal indices of $f(z)$. Then, $g(z)$ is analytic in D_R and for any z, $|z| < R$, $f(z)$ and $g(z)$ have the same maximum term. Thus, by Lemma 2.2.2, lower (α,β)-order of $g(z)$ is $\lambda(\alpha,\beta,f)$. Further, $\Psi(n_m) = |a_{n_m}/a_{n_{m+1}}|^{1/(\lambda_{n_{m+1}}-\lambda_{n_m})}$ is a strictly increasing function of m. Now, applying Theorems 2.4.3 and 2.4.4 to $g(z)$ we get

$$(2.4.35) \qquad \lambda(\alpha,\beta,f) = P(\theta(\{n_m\}))$$

and

$$(2.4.36) \qquad \lambda(\alpha,\beta,f) = P(\phi(\{n_m\})).$$

On the other hand, applying Theorems 2.4.1 and 2.4.2 to the function $f(z)$, we get

$$(2.4.37) \qquad \lambda(\alpha,\beta,f) \geq \max_{\{n_k\}} [P(\theta(\{n_k\}))]$$

and

$$(2.4.38) \qquad \lambda(\alpha,\beta,f) \geq \max_{\{n_k\}} [P(\phi(\{n_k\}))].$$

Combining (2.4.35) and (2.4.37) we get (2.4.33), while (2.4.36) and (2.4.38) give (2.4.34). This proves the theorem.

REMARKS. (i) A coefficient characterization of lower q-order, $q \geq 3$, in (1.9.6), due to Kapoor and Gopal [53], follows from (2.4.33) on taking $\alpha(x) = \log_p x$, $p \geq 2$; and $\beta(x) = \log x$; while with the same choices of $\alpha(x)$ and $\beta(x)$, (2.4.34) gives a new characterization of lower q-order for $q \geq 3$.

(ii) With $\alpha(x) = \log_p x$, $p \geq 1$, $\beta(x) = x^d$, $0 < d < \infty$, (2.4.33) improves a characterization of lower q-type, for $q \geq 3$, in (1.9.10), due to Kapoor and Gopal [53 , 55] , which was obtained under the additional restriction that the principal indices $\{\lambda_{n_m}\}$ of f(z) satisfy $\log_p \lambda_{n_m} \sim \log_p \lambda_{n_{m+1}}$ as $m \to \infty$. With the same choice of $\alpha(x)$ and $\beta(x)$, (2.4.34) gives a new characterization of lower q-type for $q \geq 3$.

(iii) For $\alpha(x) = \beta(x) = \log x$, a result analogous to (2.4.33) is due to Gopal [37] and is given by (1.9.6), with $q = 2$. Some other coefficient characterizations of lower (α,β)-order of an analytic function, for the case $\alpha(x) = \beta(x) = \log x$, are to be found in [50] and [56].

2.5. In this section we first obtain a sufficient condition on λ_n's for a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, to be of irregular (α,β)-growth. Further, a decomposition theorem for an analytic function of irregular (α,β)-growth has been proved.

THEOREM 2.5.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$ *be of* (α,β)-*order* $\rho(\alpha,\beta,f)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$. *Then if* $\alpha \not\equiv \beta$, *we have*

$$(2.5.1) \qquad \lambda(\alpha,\beta,f) \leq \rho(\alpha,\beta,f) \liminf_{n\to\infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\lambda_n)} .$$

The relation (2.5.1) *continues to hold for* $\alpha \equiv \beta$ *provided* $\lambda(\alpha,\alpha,f) > 1$.

PROOF. Under the hypothesis of the theorem, by Theorem 2.4.5, we have

$$\lambda(\alpha,\beta,f) = \max_{\{n_k\}} \left\{\liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\beta(\lambda_{n_k}/\log^+ |a_{n_k}| R^{\lambda_{n_k}})}\right\}$$

$$\leq \max_{\{n_k\}} \left\{\limsup_{k\to\infty} \frac{\alpha(\lambda_{n_k})}{\beta(\lambda_{n_k}/\log^+ |a_{n_k}| R^{\lambda_{n_k}})}\right\} \times$$

$$\times \max_{\{n_k\}} \left\{\liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\lambda_{n_k})}\right\}$$

$$= \left\{\limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta(\lambda_n/\log^+|a_n|R^{\lambda_n})}\right\} \left\{\liminf_{n\to\infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\lambda_n)}\right\}$$

$$= \rho(\alpha,\beta,f) \liminf_{n\to\infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\lambda_n)} ,$$

on using Theorem 2.3.1. This proves the theorem.

COROLLARY. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$, *be of* (α,β)-*order* $\rho(\alpha,\beta,f)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$. *We have*

(i) *If* $\alpha \not\equiv \beta$, $\liminf\limits_{n\to\infty} \alpha(\lambda_n)/\alpha(\lambda_{n+1}) < 1$ *and* $0 < \rho(\alpha,\beta,f) < \infty$ *then* $f(z)$ *is of irregular* (α,β)-*growth*.

(ii) *If* $\alpha \not\equiv \beta$, $\liminf\limits_{n\to\infty} \alpha(\lambda_n)/\alpha(\lambda_{n+1}) = 0$ *and* $\rho(\alpha,\beta,f) < \infty$, *then* $\lambda(\alpha,\beta,f) = 0$.

(iii) *If* $\alpha \not\equiv \beta$, $\liminf\limits_{n\to\infty} \alpha(\lambda_n)/\alpha(\lambda_{n+1}) = 0$ *and* $\lambda(\alpha,\beta,f) > 0$, *then* $\rho(\alpha,\beta,f) = \infty$.

(iv) *If* $\alpha \equiv \beta$, $\lambda(\alpha,\alpha,f) > 1$ *and* $\liminf\limits_{n\to\infty} \alpha(\lambda_n)/\alpha(\lambda_{n+1}) < 1$, *then* $f(z)$ *is of irregular* (α,β)-*growth*.

(v) *If* $f(z)$ *is of regular* (α,β)-*growth with* $P(0) < \rho(\alpha,\beta,f) < \infty$, *then* $\alpha(\lambda_n) \sim \alpha(\lambda_{n+1})$ *as* $n \to \infty$.

REMARK. An analogue of Theorem 2.5.1, for the case $\alpha(x) = \beta(x) = \log x$, is proved in [110] by a different technique and is given by (1.8.9).

We now prove a decomposition theorem for analytic functions of irregular (α,β)-growth.

THEOREM 2.5.2. *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$, *be of irregular* (α,β)-*growth with* (α,β)-*order* $\rho(\alpha,\beta,f)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,f)$. *Assume that* $\lambda(\alpha,\beta,f) < u < \rho(\alpha,\beta,f)$, *where* $\rho(\alpha,\beta,f) > \sigma$, $u > \sigma$, $\sigma = 0$ *if* $\alpha \not\equiv \beta$ *and* $\sigma = 1$ *if* $\alpha \equiv \beta$. *Then*

$$f(z) = g_u(z) + h_u(z),$$

where (α,β)-*order* $\rho(\alpha,\beta,g_u)$ *of* $g_u(z)$ *is atmost* u *and*

$$h_u(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}$$

<u>satisfies</u>

$$\lambda(\alpha,\beta,f) \geq u \liminf_{p\to\infty} \frac{\alpha(\lambda_{n_{p-1}})}{\alpha(\lambda_{n_p})}.$$

<u>PROOF</u>. Set $g_u(z) = \Sigma' a_n z^{\lambda_n}$, where Σ' denotes the summation over all n for which

$$(2.5.2) \qquad \log^+ |a_n| R^{\lambda_n} \leq \lambda_n/F(\lambda_n,1/u)$$

where $F(\lambda_n,1/u) = \beta^{-1}((1/u)\ \alpha(\lambda_n))$. Then $g_u(z)$ is analytic in D_R. Further, by Theorem 2.3.1, (α,β)-order $\rho(\alpha,\beta,g_u)$ of $g_u(z)$ is atmost u. Set

$$h_u(z) = f(z) - g_u(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}.$$

Then, by (2.5.2), for p = 0,1,2,..., we have

$$(2.5.3) \qquad \log^+ |a_{n_p}| R^{\lambda_{n_p}} > \lambda_{n_p}/F(\lambda_{n_p},1/u).$$

For p = 0,1,2,..., we set

$$r_p = R \exp\left(\frac{-1}{b\ F(\lambda_{n_p},1/u)}\right),$$

where b is a constant, $1 < b < \infty$. Then, for $r_p \leq r \leq r_{p+1}$, using (2.5.3) and Cauchy's inequality, we get

$$\begin{aligned}
\log M(r,f) &\geq \log |a_{n_p}| + \lambda_{n_p} \log r \\
&\geq \log |a_{n_p}| R^{\lambda_{n_p}} + \lambda_{n_p} \log (r_p/R) \\
&\geq \lambda_{n_p}/F(\lambda_{n_p},1/u) + \lambda_{n_p} \log (r_p/R) \\
&= (b-1)\ \lambda_{n_p} \log (R/r_p) \\
&= (b-1)\ \lambda_{n_p}/(b\ F(\lambda_{n_p},1/u)).
\end{aligned}$$

The above relation, on applying (2.1.3), gives that

$$\frac{\alpha(\frac{b}{b-1}\log M(r,f))}{\beta(1/\log(R/r))} \geq \frac{\alpha(\lambda_{n_p}/F(\lambda_{n_p},1/u))}{\beta(1/\log(R/r_{p+1}))}$$

$$\sim \frac{\alpha(\lambda_{n_p})}{\beta(1/\log(R/r_{p+1}))}$$

$$= \frac{\alpha(\lambda_{n_p})}{\beta(b\ F(\lambda_{n_{p+1}},\ 1/u))}$$

$$= u\ \frac{\alpha(\lambda_{n_p})}{\alpha(\lambda_{n_{p+1}})}\ \frac{\beta(F(\lambda_{n_{p+1}},\ 1/u))}{\beta(b\ F(\lambda_{n_{p+1}},1/u))}\ .$$

Since $\alpha(x)\ \varepsilon\ \Lambda$ and $\beta(x)\ \varepsilon\ L^{o}$, using Lemma 2.2.1, the above relation yields that

$$\lambda(\alpha,\beta,f) \geq u \liminf_{p\to\infty} \frac{\alpha(\lambda_{n_p})}{\alpha(\lambda_{n_{p+1}})}\ .$$

This proves the theorem.

REMARKS. (i) Taking $\alpha(x) = \log_p x$, $p \geq 2$, and $\beta(x) = \log x$ or $\alpha(x) = \log_p x$, $p \geq 1$, and $\beta(x) = x^d$, $0 < d < \infty$, some results of Kapoor and Gopal [53, Theorems 4 and 10] follow from the above theorem.

(ii) For the case $\alpha(x) = \beta(x) = \log x$, a result analogous to Theorem 2.5.2 was obtained by Sons [110] :

CHAPTER 3

GENERALIZED ORDERS OF FUNCTIONS ANALYTIC IN A FINITE DISC II

3.1. Let $f(z)$, analytic in D_R, $0 < R < \infty$, be of slow growth, i.e., order $\rho(f)$ of $f(z)$, given by (1.8.1), is zero. Then, for any choice of $\alpha(x)$ and $\beta(x)$ satisfying (2.1.3), the (α,β)-order $\rho(\alpha,\beta,f)$ of $f(z)$ satisfies $\rho(\alpha,\beta,f) = 0$ if $\alpha \not\equiv \beta$ and $\rho(\alpha,\beta,f) \leq 1$ if $\alpha \equiv \beta$. Thus, for such functions, concepts introduced in Section 2.1 and investigated in Sections 2.2 to 2.5 do not give distinct (α,β)-growth parameters for $\alpha \not\equiv \beta$. Further, when $\alpha \equiv \beta$, the results obtained in Sections 2.2 to 2.5 either fail to give any specific information or are not applicable to the functions of slow growth. This difficulty in measuring the growth of analytic functions is taken care of by the concepts of α-logarithmic order, lower α-logarithmic order, (α,α_*)-logarithmic type and lower (α,α_*)-logarithmic type introduced below. We have

<u>DEFINITION</u> 3.1.1. <u>A function</u> $f(z)$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>is said to be of</u> α-<u>logarithmic order</u> $\rho(\alpha,f)$ <u>and lower</u> α-<u>logarithmic order</u> $\lambda(\alpha,f)$, $0 \leq \lambda(\alpha,f) \leq \rho(\alpha,f) \leq \infty$, <u>if</u>

$$\begin{matrix}\rho(\alpha,f)\\ \\ \lambda(\alpha,f)\end{matrix} = \lim_{r \to R} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\alpha(\log (R/(R-r)))},$$

<u>where</u> $\alpha(x) \in \Lambda$ <u>and for all</u> c, $1 \leq c < \infty$, $G(x,c) = \alpha^{-1}(c\,\alpha(\log x))$ <u>satisfies</u>

(3.1.1) $G(x,c)/x$ is ultimately a nonincreasing function of x an

$$\lim_{x \to \infty} G(x,c)/x = 0.$$

DEFINITION 3.1.2. A function $f(z)$, analytic in D_R, $0 < R < \infty$, with α-logarithmic order $\rho(\alpha,f)$ and lower α-logarithmic order $\lambda(\alpha,f)$ is said to be of regular α-logarithmic growth if $\lambda(\alpha,f) = \rho(\alpha,f)$. The function $f(z)$ is said to be of irregular α-logarithmic growth if it is not of regular α-logarithmic growth

Taking, in particular, $\alpha(x) = \log x$, $\rho(\alpha,f)$ and $\lambda(\alpha,f)$ reduce, respectively, to the logarithmic order and lower logarithmic order of an analytic function, introduced in Kapoor and Gopal [54] and [37].

The following are some other choices of $\alpha(x)$ which also satisfy the hypotheses in Definition 3.1.1.

(i) $\alpha(x) = \log_p x$, $p \geq 1$

(ii) $\alpha(x) = \exp (\log x)^{\ell}$, $0 < \ell < 1$

(iii) $\alpha(x) = \exp (\log_p x)^{\ell}$, $p \geq 2$, $0 < \ell < \infty$.

DEFINITION 3.1.3. Let $f(z)$, analytic in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,f)$ with $1 < \rho(\alpha,f) < \infty$. Then, $f(z)$ is said to be of (α,α_*)-logarithmic type $T(\alpha,\alpha_*,f)$ and lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$ if

$$(3.1.2) \qquad \begin{matrix} T(\alpha,\alpha_*,f) \\ t(\alpha,\alpha_*,f) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha_*(\log M(r,f))}{(\alpha_*(\log (R/(R-r))))^{\rho(\alpha,f)}},$$

where $\alpha_*(x) \in L^0$ and for $0 < c < \infty$ and $1 < d < \infty$, $G(x,c,d) = \alpha_*^{-1}(c(\alpha_*(\log x))^d)$ satisfies

(3.1.3) $G(x,c,d)/x$ *is ultimately a nonincreasing function of* *and*

(3.1.4) $\log G(x,c,d)/\log x \to 0$ *as* $x \to \infty$.

Further, if $\alpha_*(x) \in \Lambda$ *we assume that a weaker condition*

(3.1.5) $G(x,c,d)/x \to 0$ *as* $x \to \infty$

holds in place of the condition (3.1.4).

Taking $\alpha(x) = \log x$ and $\alpha_*(x) = x$, $T(\alpha,\alpha_*,f)$ and $t(\alpha,\alpha_*,f)$ reduce, respectively, to the logarithmic type and lower logarithmic type of an analytic function, introduced in [54] and [37].

The following choices of $\alpha_*(x)$ also satisfy the hypotheses in Definition 3.1.3.

(i) $\alpha_*(x) = \log_p x$, $p \geq 0$.

(ii) $\alpha_*(x) = \exp(\log x)^{\ell}$, $0 < \ell < 1$.

We observe that the concepts introduced in Definitions 3.1.1 and 3.1.3 give a satisfactory measure for functions analytic in D_R, $0 < R < \infty$, having slow rates of growth.

In Section 3.2, characterizations of lower α-logarithmic order $\lambda(\alpha,f)$ of a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, have been found in terms of the coefficients a_n. A decomposition theorem for functions of irregular α-logarithmic growth is also proved in this section. In Section 3.3, we first obtain a complete coefficient characterization of (α,α_*)-logarithmic type. The coefficient characterizations of lower

(α,α_*)-logarithmic type are also found. Finally, in the same section, a decomposition theorem is proved for an analytic function $f(z)$ with α-logarithmic order $\rho(\alpha,f)$, $1 < \rho(\alpha,f) < \infty$, (α,α_*)-logarithmic type $T(\alpha,\alpha_*,f)$ and lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$ such that $t(\alpha,\alpha_*,f) < T(\alpha,\alpha_*,f)$. Our results in this chapter offer wide generalizations of the results in [54] and [37].

3.2. For a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$ having α-logarithmic order $\rho(\alpha,f)$, set

$$(3.2.1) \qquad \theta_\alpha = \limsup_{n\to\infty} \frac{\alpha(\log |a_n| R^{\lambda_n})}{\alpha(\log \lambda_n)} .$$

Then, the techniques of Šeremata [92] can be adopted to obtain

$$(3.2.2) \qquad \theta_\alpha \le \rho(\alpha,f) \le \max(1,\theta_\alpha).$$

However, a relation analogous to (3.2.2) is not known for lower α-logarithmic order. Thus, in Theorems 3.2.1 to 3.2.4 we investigate interrelations between the lower α-logarithmic order $\lambda(\alpha,f)$ and the Taylor coefficients of an analytic function $f(z)$. Theorem 3.2.5 is a decomposition theorem for analytic functions of irregular α-logarithmic growth.

<u>THEOREM</u> 3.2.1. <u>*Let*</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ <u>*be analytic in*</u> D_R, $0 < R < \infty$, <u>*and let*</u> $\alpha(x) \in \Lambda$. <u>*Then*</u>

$$(3.2.3) \qquad \liminf_{r\to R} \frac{\alpha(\log M(r,f))}{\alpha(\log (R/(R-r)))} \ge \liminf_{k\to\infty} \frac{\alpha(\log |a_{n_k}| R^{\lambda_{n_k}})}{\alpha(\log \lambda_{n_{k+1}})} ,$$

<u>for any increasing sequence</u> $\{n_k\}$ <u>of positive integers</u>.

<u>PROOF</u>. Let the limit inferior on the right hand side of (3.2.3) be denoted by θ_α. Clearly $0 \leq \theta_\alpha \leq \infty$. First, let $0 < \theta_\alpha < \infty$. Then, given $\varepsilon > 0$, $\theta_\alpha > \varepsilon$, there exists $k_o = k_o(\varepsilon)$ such that, for all $k > k_o$, we have

(3.2.4) $$\log |a_{n_k}| R^{\lambda_{n_k}} > G(\lambda_{n_{k+1}}, \bar{\theta}_\alpha)$$

where $G(\lambda_{n_{k+1}}, \bar{\theta}_\alpha) = \alpha^{-1}(\bar{\theta}_\alpha\, \alpha(\log \lambda_{n_{k+1}}))$ and $\bar{\theta}_\alpha = \theta_\alpha - \varepsilon$. For $k > k_o$, we choose

$$r_k = R(1-1/\lambda_{n_k}).$$

For $r_k \leq r \leq r_{k+1}$, using (3.2.4) and Cauchy's inequality, we get

$$\begin{aligned}\log M(r,f) &\geq \log |a_{n_k}| R^{\lambda_{n_k}} + \lambda_{n_k} \log (r/R)\\ &\geq G(\lambda_{n_{k+1}}, \bar{\theta}_\alpha) + \lambda_{n_k} \log (r_k/R)\\ &= G(R/(R-r_{k+1}), \bar{\theta}_\alpha) + \lambda_{n_k} \log (1-1/\lambda_{n_k})\\ &\geq G(R/(R-r), \bar{\theta}_\alpha) + O(1).\end{aligned}$$

Since $\alpha(x) \in \Lambda$, the above relation gives that

$$\liminf_{r \to R} \frac{\alpha(\log M(r,f))}{\alpha(\log R/(R-r)))} \geq \bar{\theta}_\alpha = \theta_\alpha - \varepsilon,$$

and so, letting $\varepsilon \to 0$, we obtain

(3.2.5) $$\liminf_{r \to R} \frac{\alpha(\log M(r,f))}{\alpha(\log (R/(R-r)))} \geq \theta_\alpha.$$

Obviously, (3.2.5) holds for $\theta_\alpha = 0$. For $\theta_\alpha = \infty$, the above analysis with an arbitrarily large number in place of $\bar{\theta}_\alpha$ gives that the limit inferior on the left hand side of (3.2.5) is also infinite. This proves the theorem.

COROLLARY. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D_R, $0 < R < \infty$, *with* α-*logarithmic order* $\rho(\alpha, f)$ *and lower* α-*logarithmic order* $\lambda(\alpha, f)$. *Further, assume that*

$$\text{(i)} \qquad \lim_{n \to \infty} \alpha(\log \lambda_n)/\alpha(\log \lambda_{n+1}) = 1$$

and

$$\text{(ii)} \qquad \lim_{n \to \infty} \frac{\alpha(\log |a_n| R^{\lambda_n})}{\alpha(\log \lambda_n)} = S_0 \text{ exists with } 1 < S_0 < \infty.$$

Then, $f(z)$ *is of regular* α-*logarithmic growth with* $\rho(\alpha, f) = \lambda(\alpha, f) = S_0$.

The corollary is immediate in view of (3.2.2) and Theorem 3.2.1 and so the proof is omitted.

REMARKS. (i) With $\alpha(x) = \log x$, a result of Gopal [37 , Theorem 4.3.1] follows from the above theorem.

(ii) We now give some examples which show that the concepts of α-logarithmic order and lower α-logarithmic order introduced here are more refined than the concepts of logarithmic order and lower logarithmic order introduced in [54] and [37] .

EXAMPLE. Let

$$F_1(z) = \sum_{n=3}^{\infty} \exp\exp\,(\log_2 n)^K\,(z/R)^n,\ 1 < K < \infty, 0 < R < \infty,$$

Then $F_1(z)$ is analytic in D_R and the order $\rho(F_1)$ of $F_1(z)$, given by (1.8.1), is zero. From Theorem 3.2.1 it follows that $\lambda(\log, F_1) = \rho(\log, F_1) = \infty$. Thus, the growth parameters in ([54], [37]) fail to give any specific information about the growth of $F_1(z)$. However, from the corollary of Theorem 3.2.1, we have $\rho(\log_2, F_1) = \lambda(\log_2, F_1) = K$ and so the function $F_1(z)$ has nonzero finite growth parameters in our sense with the choice $\alpha(x) = \log_2 x$.

<u>EXAMPLE</u>. Let $\alpha_\ell(x) = \exp\,(\log x)^\ell$, $0 < \ell < 1$. For a fixed ℓ and a fixed R, $0 < R < \infty$, we consider a family of functions $\mathcal{F}_\ell$ defined by

$$\mathcal{F}_\ell = \{ f_K : f_K(z) = \sum_{n=3}^{\infty} \exp(\alpha_\ell^{-1}(K\,\alpha_\ell(\log n)))(z/R)^n,\ 1 < K < \infty \}.$$

Then, each member of $\mathcal{F}_\ell$ is analytic in D_R and is of order zero. Further, for each $f_K \in \mathcal{F}_\ell$, by (3.2.2) and Theorem 3.2.1 we have $\rho(\log, f_K) = \lambda(\log, f_K) = 1$. Thus, confining to the growth parameters in ([54], [37]), we can not compare the growths of any two members of the family $\mathcal{F}_\ell$. However, with the choice $\alpha(x) = \alpha_\ell(x)$, we see, from the corollary of Theorem 3.2.1, that $\rho(\alpha_\ell, f_K) = \lambda(\alpha_\ell, f_K) = K$ for every $f_K \in \mathcal{F}_\ell$. Thus, our growth scheme assigns distinct growth parameters to every $f_K \in \mathcal{F}_\ell$.

The following lemma is needed in the proof of our next theorem.

<u>LEMMA</u> 3.2.1. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>be of</u> α-<u>logarithmic order</u> $\rho(\alpha, f)$ <u>and lower</u> α-<u>logarithmic order</u>

$\lambda(\alpha, f)$. *Set*

$$\begin{matrix}\Phi_\alpha \\ \phi_\alpha\end{matrix} = \lim_{r \to R} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log \mu(r))}{\alpha(\log (R/(R-r)))}.$$

Then, if $\rho(\alpha, f) > 1$, *we have* $\rho(\alpha, f) = \Phi_\alpha$. *Further, if* $\lambda(\alpha, f) >$ *then* $\lambda(\alpha, f) = \phi_\alpha$ *also holds*.

PROOF. The hypothesis $\rho(\alpha, f) > 1$ implies that $\mu(r)$ is an unbounded function of r, since otherwise the sequence $\{|a_n| R^{\lambda n}\}$ is bounded and so $\rho(\alpha, f) \leq 1$. Now, by (2.2.13), with $b = 3$, we have

$$(3.2.6) \quad \log M(r,f) \leq \log \mu(r) + 2\log \frac{R}{R-r} + \log 3 + \log \frac{9}{2}$$
$$+ \log \log \mu(r + 2\frac{R-r}{3})$$
$$\leq \max \{5 \log \mu(r + 2\frac{R-r}{3}), 5 \log \frac{R}{R-r}\}$$

for all r sufficiently near to R. Hence, for all r sufficiently near to R, we get

$$\alpha(\log M(r,f)) \leq \max \{\alpha(5 \log \mu(r+2\frac{R-r}{3})), \alpha(5 \log \frac{R}{R-r})\},$$

and so, since $\alpha(x) \in \Lambda$, we obtain

$$(3.2.7) \qquad \rho(\alpha, f) \leq \max (\Phi_\alpha, 1), \lambda(\alpha, f) \leq \max (\phi_\alpha, 1).$$

On the other hand, the well known inequality $\mu(r) \leq M(r,f)$ gives that

$$(3.2.8) \qquad \Phi_\alpha \leq \rho(\alpha, f), \phi_\alpha \leq \lambda(\alpha, f).$$

The lemma now follows immediately from (3.2.7) and (3.2.8).

<u>THEOREM</u> 3.2.2. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>be of lower</u> α-<u>logarithmic order</u> $\lambda(\alpha, f)$ <u>with</u> $\lambda(\alpha, f) > 1$. <u>Assume that, for all</u> c, $1 \leq c < \infty$, $G(x,c) = \alpha^{-1}(c\, \alpha(\log x))$ <u>satisfies</u>

$$\frac{dG(x,c)}{dx} \leq G(x,c)/x \quad \text{for} \quad x > x_o(c) \tag{3.2.9}$$

<u>and</u>

$$x^2 \frac{dG(x,c)}{dx} \to \infty \quad \text{as } x \to \infty. \tag{3.2.10}$$

<u>Further, if</u> $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ <u>is a nondecreasing function of</u> n <u>for</u> $n > n_o$, <u>then</u>

$$\lambda(\alpha, f) \leq \liminf_{n \to \infty} \frac{\alpha(\log |a_n| R^{\lambda_n})}{\alpha(\log \lambda_n)} .$$

<u>PROOF</u>. Clearly $\psi(n) > \psi(n-1)$ for infinitely many values of n, since otherwise the sequence $\{|a_n| R^{\lambda_n}\}$ is bounded and so $\rho(\alpha, f) \leq 1$, which is false because $\lambda(\alpha, f) > 1$. Further, $\Psi(n) \to R$ as $n \to \infty$. When $\psi(n) > \Psi(n-1)$, the term $a_n z^{\lambda_n}$ becomes the maximum term and we have

$$\mu(r) = |a_n| r^{\lambda_n} \text{ for } \Psi(n-1) \leq r < \Psi(n).$$

Now, by Lemma 3.2.1, we have

$$\lambda(\alpha, f) = \liminf_{r \to R} \frac{\alpha(\log \mu(r))}{\alpha(\log (R/(R-r)))} .$$

We first assume that $\lambda(\alpha, f) < \infty$. Then, given $\varepsilon > 0$, there exists $r_o = r_o(\varepsilon)$ such that, for $r > r_o$, we have

$$\log \mu(r) > G(R/(R-r), \bar{\lambda}_\alpha)$$

where $G(R/(R-r), \bar{\lambda}_\alpha) = \alpha^{-1}(\bar{\lambda}_\alpha \alpha(\log (R/(R-r))))$ and $\bar{\lambda}_\alpha = \lambda(\alpha, f) - \varepsilon >$

Let $a_{n_1} z^{\lambda_{n_1}}$ and $a_{n_2} z^{\lambda_{n_2}}$ $(n_1 > n_o, \Psi(n_1-1) > r_o)$ be two consecutive maximum terms so that $n_1 \le n_2-1$. Then

$$\log |a_{n_2}| + \lambda_{n_2} \log r > G(R/(R-r), \bar{\lambda}_\alpha)$$

for every r satisfying $\Psi(n_2-1) \le r < \Psi(n_2)$. Let $n_1 \le n \le n_2-1$. Then $\Psi(n_1) = \Psi(n_1+1) = \dots = \Psi(n) = \dots = \Psi(n_2-1)$ and

$$|a_n| r^{\lambda_n} = |a_{n_2}| r^{\lambda_{n_2}} \text{ for } r = \Psi(n).$$

Hence

$$\log |a_n| + \lambda_n \log \Psi(n) \ge G(R/(R-\Psi(n)), \bar{\lambda}_\alpha)$$

or

$$\log |a_n| R^{\lambda_n} \ge \lambda_n \log (R/\Psi(n)) + G(R/(R-\Psi(n)), \bar{\lambda}_\alpha) \tag{3.2.11}$$

$$\ge \lambda_n (R-\Psi(n))/R + G(R/(R-\Psi(n)), \bar{\lambda}_\alpha).$$

Now, consider the function

$$H(x) = \lambda_n/x + G(x, \bar{\lambda}_\alpha).$$

Differentiating $H(x)$, we get

$$H'(x) = -\lambda_n/x^2 + \frac{dG(x, \bar{\lambda}_\alpha)}{dx}. \tag{3.2.12}$$

By the hypothesis of the theorem, we now have $G(x, \bar{\lambda}_\alpha) < x$ and $dG(x, \bar{\lambda}_\alpha)/dx \le G(x, \bar{\lambda}_\alpha)/x$ for $x \ge x^o = x^o(\bar{\lambda}_\alpha)$. Thus

$$H'(x^o) = -\lambda_n/(x^o)^2 + \left.\frac{dG(x, \bar{\lambda}_\alpha)}{dx}\right|_{x=x^o} < 0$$

for $n > n^{o} = n^{o}(x^{o})$. On the other hand, since (3.2.10) holds, by (3.2.12), we have $H'(x) > 0$ for all sufficiently large value of x. We now define, for $n > n^{o}$, $x_*(n)$ as

$$H(x_*(n)) = \min_{x^{o} < x < \infty} H(x).$$

Then

$$\frac{\lambda_n}{(x_*(n))^2} = \frac{dG(x, \bar{\lambda}_\alpha)}{dx}\bigg|_{x=x_*(n)}$$

and so

(3.2.13) $$\lambda_n \leq x_*(n)\ G(x_*(n), \bar{\lambda}_\alpha) \leq (x_*(n))^2.$$

From (3.2.11), (3.2.13) and the definition of $x_*(n)$, for all sufficiently large values of n, we obtain

$$\log |a_n|\ R^{\lambda_n} \geq G(\sqrt{\lambda_n}, \bar{\lambda}_\alpha)$$

or

$$\frac{\alpha(\log |a_n|\ R^{\lambda_n})}{\alpha(\frac{1}{2} \log \lambda_n)} \geq \bar{\lambda}_\alpha = \lambda(\alpha, f) - \varepsilon.$$

Since $\alpha(x) \in \Lambda$ and $\varepsilon > 0$ is arbitrary, the above inequality gives that

(3.2.14) $$\liminf_{n \to \infty} \frac{\alpha(\log |a_n|\ R^{\lambda_n})}{\alpha(\log \lambda_n)} \geq \lambda(\alpha, f).$$

If $\lambda(\alpha, f) = \infty$, the above arguments with an arbitrarily large number in place of $\bar{\lambda}_\alpha$ give that the limit inferior on the left hand side of (3.2.14) is also ∞.

This proves the theorem.

REMARKS. (i) The condition (3.2.9) implies that $G(x,c)/x$ is ultimately a nonincreasing function of x for every c, $1 \leq c < \infty$.

(ii) In particular, $\alpha(x) = \log_p x, p \geq 1$, satisfies the conditions (3.2.9) and (3.2.10).

Combining Theorems 3.2.1 and 3.2.2 we obtain a coefficient characterization of lower α-logarithmic order $\lambda(\alpha,f)$ of a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$. This is our Theorem 3.2.3.

THEOREM 3.2.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, be of lower α-logarithmic order $\lambda(\alpha,f)$ with $\lambda(\alpha,f) > 1$. Assume that (3.2.9) and (3.2.10) are satisfied. Further, let $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ be ultimately a nondecreasing function of n and let $\alpha(\log \lambda_n) \sim \alpha(\log \lambda_{n+1})$ as $n \to \infty$. Then*

$$\lambda(\alpha,f) = \liminf_{n \to \infty} \frac{\alpha(\log |a_n| R^{\lambda_n})}{\alpha(\log \lambda_{n+1})} .$$

Our next theorem gives a coefficient characterization of lower α-logarithmic order which does not require the nondecreasing nature of the function $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$.

THEOREM 3.2.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, be of lower α-logarithmic order $\lambda(\alpha,f)$ with $\lambda(\alpha,f) \geq 1$. Assume that (3.2.9) and (3.2.10) are satisfied. Further, let $\alpha(\log \lambda_{n_m}) \sim \alpha(\log \lambda_{n_{m+1}})$ as $m \to \infty$, where $\{\lambda_{n_m}\}$ is the sequence of the principal indices of $f(z)$. Then*

(3.2.15) $$\lambda(\alpha, f) = \max(1, \theta_\alpha^*)$$

<u>where</u>

(3.2.16) $$\theta_\alpha^* = \max_{\{n_k\}} \left\{ \liminf_{k \to \infty} \frac{\alpha(\log |a_{n_k}| R^{\lambda_{n_k}})}{\alpha(\log \lambda_{n_{k+1}})} \right\},$$

<u>and the maximum in</u> (3.2.16) <u>is taken over all increasing sequenc</u> $\{n_k\}$ <u>of positive integers</u>.

<u>PROOF</u>. First, let $\lambda(\alpha, f) = 1$. Then θ_α^*, given by (3.2.16), is atmost one, by Theorem 3.2.1. Thus, (3.2.15) holds in this case

Next, let $\lambda(\alpha, f) > 1$. We consider the function $g(z) = \sum_{m=0}^{\infty} a_{n_m} z^{\lambda_{n_m}}$, where $\{\lambda_{n_m}\}$ is the sequence of the principal indic of $f(z)$. Then $g(z)$ is analytic in D_R and for any z, $|z| < R$, $f(z)$ and $g(z)$ have the same maximum term. Thus, by Lemma 3.2.1, lower α-logarithmic order of $g(z)$ is $\lambda(\alpha, f)$. Also, $\Psi(n_m) = |a_{n_m}/a_{n_{m+1}}|^{1/(\lambda_{n_{m+1}} - \lambda_{n_m})}$ is a strictly increasing function of m. Now, since $g(z)$ satisfies the hypothesis of Theorem 3.2.3, we have

(3.2.17) $$\lambda(\alpha, f) = \liminf_{m \to \infty} \frac{\alpha(\log |a_{n_m}| R^{\lambda_{n_m}})}{\alpha(\log \lambda_{n_{m+1}})}.$$

Further, by Theorem 3.2.1, we have

(3.2.18) $$\lambda(\alpha, f) \geq \max_{\{n_k\}} \left\{ \liminf_{k \to \infty} \frac{\alpha(\log |a_{n_k}| R^{\lambda_{n_k}})}{\alpha(\log \lambda_{n_{k+1}})} \right\}.$$

From (3.2.17) and (3.2.18) we get (3.2.15) for the case $\lambda(\alpha, f) > 1$ also.

This completes the proof of the theorem.

REMARK. With $\alpha(x) = \log x$, a result of Gopal [37, Theorem 4.: follows from the above theorem.

We now prove a decomposition theorem for analytic functions of irregular α-logarithmic growth.

THEOREM 3.2.5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$ *be of irregular* α-*logarithmic growth with* α-*logarithmic order* $\rho(\alpha,f)$, $\rho(\alpha,f) > 1$, *and lower* α-*logarithmic order* $\lambda(\alpha,f)$. *If* $\lambda(\alpha,f) < u < \rho(\alpha,f)$, *then*

$$f(z) = g_u(z) + h_u(z)$$

where α-*logarithmic order of* $g_u(z)$ *is atmost* max $(1,u)$ *and*

$$h_u(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda n_p}$$

satisfies

$$\lambda(\alpha,f) \geq u \liminf_{p \to \infty} \frac{\alpha(\log \lambda_{n_p})}{\alpha(\log \lambda_{n_{p+1}})} .$$

PROOF. Let $g_u(z) = \Sigma' a_n z^{\lambda_n}$, where Σ' denotes the summation over n for which

$$(3.2.19) \qquad |a_n| R^{\lambda_n} \leq \exp(G(\lambda_n,u))$$

where $G(\lambda_n,u) = \alpha^{-1}(u\,\alpha(\log \lambda_n))$. Then $g_u(z)$ is analytic in D_R and, by (3.2.2), α-logarithmic order of $g_u(z)$ is atmost max $(1,u)$

Set

$$h_u(z) = f(z) - g_u(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}},$$

Then, by (3.2.19), we get

$$(3.2.20) \qquad |a_{n_p}|\, R^{\lambda_{n_p}} > \exp\,(G(\lambda_{n_p},u)).$$

For $p = 0,1,2,\ldots$, let $r_p = R(1-1/\lambda_{n_p})$. For $r_p \le r \le r_{p+1}$, using Cauchy's inequality and (3.2.20), we have

$$\begin{aligned}\log M(r,f) &\ge \log |a_{n_p}| + \lambda_{n_p} \log r \\ &\ge \log |a_{n_p}|\, R^{\lambda_{n_p}} + \lambda_{n_p} \log (r_p/R) \\ &\ge G(\lambda_{n_p},u) + \lambda_{n_p} \log (1-1/\lambda_{n_p}) \\ &\ge G(\lambda_{n_p},u)/2,\end{aligned}$$

for all sufficiently large values of p. Thus

$$\begin{aligned}\frac{\alpha(2 \log M(r,f))}{\alpha(\log (R/(R-r)))} &\ge u \frac{\alpha(\log \lambda_{n_p})}{\alpha(\log (R/(R-r_{p+1})))} \\ &= u \frac{\alpha(\log \lambda_{n_p})}{\alpha(\log \lambda_{n_{p+1}})},\end{aligned}$$

for all r sufficiently near to R. Since $\alpha(x) \in \Lambda$, the above relation, on passing to limits, gives that

$$\lambda(\alpha,f) = \liminf_{r \to R} \frac{\alpha(2 \log M(r,f))}{\alpha(\log (R/(R-r)))} \ge u \liminf_{p \to \infty} \frac{\alpha(\log \lambda_{n_p})}{\alpha(\log \lambda_{n_{p+1}})}.$$

This completes the proof of the theorem.

<u>REMARK</u>. Taking, in particular, $\alpha(x) = \log x$, the above theorem generalizes a result in [54].

3.3. Coefficient characterization of (α,α_*)-logarithmic type $T(\alpha,\alpha_*,f)$ of a function $f(z)$, analytic in D_R, $0 < R < \infty$, has been found in Theorem 3.3.1; while the coefficient characterizations of lower (α,α_*)-logarithmic type $t(\alpha;\alpha_*;f)$ of $f(z)$ are to be found in Theorems 3.3.2 and 3.3.3. Finally, in Theorem 3.3.4, we obtain a decomposition theorem.

We need the following lemma.

LEMMA 3.3.1. *If* $\alpha(x) \varepsilon L^0$, *then, for* $0 < c < \infty$ *and* $1 < d < \infty$, *we have*

$$\frac{x}{\alpha^{-1}(c(\alpha(x))^d)} \to 0 \text{ as } x \to \infty. \tag{3.3.1}$$

PROOF. Suppose, on the contrary, that (3.3.1) does not hold. Then there exist some c, $0 < c < \infty$, and d, $1 < d < \infty$, such that for a sequence $\{x_k\}$, $x_k \to \infty$, we have

$$x_k > M\, \alpha^{-1}(c(\alpha(x_k))^d),$$

where $M > 0$. This gives

$$\frac{\alpha(x_k/M)}{(\alpha(x_k))^d} > c \tag{3.3.2}$$

for the sequence $\{x_k\}$. Now, since $\alpha(x) \varepsilon L^0$, by Lemma 2.2.1, there exists ℓ, $0 < \ell < 1$, such that

$$\liminf_{x \to \infty} \frac{\alpha(\ell x)}{\alpha(x)} \geq 1/2$$

and so there exists $x_* = x_*(\ell)$ such that, for $x \geq x_*$, we have

$$\frac{\alpha(\ell x)}{\alpha(x)} \geq 1/4$$

or, equivalently,

$$\alpha(x/\ell) \leq 4\ \alpha(x)$$

for $x \geq x_* \ell$. Since $M > C$ and $\ell < 1$, we can choose a positive integer N such that $M \geq \ell^N$. Thus, for $x \geq x_* \ell$, we have

$$\alpha(x/M) \leq \alpha(x/\ell^N) \leq 4\ \alpha(x/\ell^{N-1}) \leq \dots \leq 4^N\ \alpha(x). \tag{3.3.3}$$

Now, it is easily seen that, in view of (3.3.3), (3.3.2) is false. Thus, our supposition that (3.3.1) does not hold is not true. This proves the lemma.

<u>THEOREM</u> 3.3.1. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>be of</u> α-<u>logarithmic order</u> $\rho(\alpha, f)$, $1 < \rho(\alpha, f) < \infty$, <u>and</u> (α, α_*)-<u>logarithmic type</u> $T(\alpha, \alpha_*, f)$. <u>Then</u>

$$T(\alpha, \alpha_*, f) = \limsup_{n \to \infty} \frac{\alpha_*(\log |a_n|\ R^{\lambda_n})}{(\alpha_*(\log \lambda_n))^{\rho(\alpha, f)}}. \tag{3.3.4}$$

<u>PROOF</u>. First, let $T(\alpha, \alpha_*, f) < \infty$. Then, for $r > r_o = r_o(\varepsilon)$, $\varepsilon > 0$, by (3.1.2), we have

$$\log M(r, f) < \alpha_*^{-1}\ (T'(\alpha_*(\log\ (R/(R-r))))^{\rho(\alpha, f)})$$

where $T' = T(\alpha, \alpha_*, f) + \varepsilon$. We define a sequence $\{r_n\}$ by $r_n = R(1-1/\lambda_n)$. Using Cauchy's inequality and the above bound of $\log M(r, f)$, for all sufficiently large values of n, we have

$$\alpha_*^{-1}(T'(\alpha_*(\log \lambda_n))^{\rho(\alpha, f)}) \geq \log |a_n|\ R^{\lambda_n} + \lambda_n \log\ (r_n/R)$$

$$= \log |a_n|\ R^{\lambda_n} + \lambda_n \log\ (1-1/\lambda_n).$$

We observe that $\lambda_n \log (1 - \frac{1}{\lambda_n}) \to -1$ as $n \to \infty$. Now, since $\alpha_*(x) \in L^0$ and $\varepsilon > 0$ is arbitrary, the above relation gives that

$$(3.3.5) \qquad T(\alpha,\alpha_*,f) \geq \limsup_{n \to \infty} \frac{\alpha_*(\log |a_n| R^{\lambda_n})}{(\alpha_*(\log \lambda_n))^{\rho(\alpha,f)}} .$$

Obviously, (3.3.5) holds for $T(\alpha,\alpha_*,f) = \infty$.

Now, let the limit superior on the right hand side of (3.3.4) be denoted by θ_*. Clearly $0 \leq \theta_* \leq \infty$. First, let $\theta_* < \infty$. Then, for $n > n_0 = n_0(\varepsilon)$, $\varepsilon > 0$, we have

$$(3.3.6) \qquad \log |a_n| R^{\lambda_n} \leq G(\lambda_n, \theta_*', \rho(\alpha,f))$$

where $G(\lambda_n, \theta_*', \rho(\alpha,f)) = \alpha_*^{-1}(\theta_*'(\alpha_*(\log \lambda_n))^{\rho(\alpha,f)})$ and $\theta_*' = \theta_*+\varepsilon$.

Thus

$$(3.3.7) \qquad M(r,f) \leq \sum_{n=0}^{\infty} |a_n| r^{\lambda_n}$$

$$\leq A(\lambda_{n_0}) + \sum_{n=n_0+1}^{\infty} \exp(G(\lambda_n,\theta_*', \rho(\alpha,f))) (r/R)^{\lambda_n},$$

where $A(\lambda_{n_0})$, the sum of first n_0+1 terms, is bounded. For every $r(< R)$ sufficiently near to R, we find a natural number $n(r)$ such that

$$(3.3.8) \qquad \lambda_{n(r)} \leq \frac{2}{\log (R/r)} G(2/\log (R/r), 2\theta_*',\rho(\alpha,f)) < \lambda_{n(r)+1}.$$

Then, by (3.1.4) and (3.1.5), we get, for all r sufficiently near to R,

$$(3.3.9) \qquad \log \{\frac{2}{\log (R/r)} G(2/\log (R/r), 2\,\theta_*', \rho(\alpha,f))\}$$

$$\leq \begin{cases} 2 \log (2/\log (R/r)) \text{ if } \alpha_*(x) \in \Lambda \\ (1+o(1))\log(2/\log(R/r)) \text{ if } \alpha_*(x) \in L^0 \text{ and } \alpha_*(x) \notin \Lambda . \end{cases}$$

Now, for $n > n(r)$, r sufficiently near to R, by (3.1.3), we have

$$\frac{G(\lambda_n,\theta_*',\rho(\alpha,f))}{\lambda_n} \leq \frac{G(\lambda_{n(r)+1},\theta_*',\rho(\alpha,f))}{\lambda_{n(r)+1}}$$

$$\leq \frac{G(\frac{2}{\log (R/r)}G(2/\log(R/r),2\theta_*',\rho(\alpha,f)),\theta_*',\rho(\alpha,f))}{\frac{2}{\log (R/r)}\ G(2/\log (R/r),2\ \theta_*',\ \rho(\alpha,f))}$$

$$\leq (\log (R/r))/2,$$

on using (3.3.9). Thus, for all r sufficiently near to R, we get

$$(3.3.10) \qquad \sum_{n=n(r)+1}^{\infty} |a_n| r^{\lambda_n} \leq \sum_{n=n(r)+1}^{\infty} (r/R)^{\lambda_n/2} \leq \frac{1}{1-\sqrt{r/R}}\ .$$

By (3.3.6), (3.3.7), (3.3.10) and the definition of $n(r)$, we now have

$$(3.3.11) \quad M(r,f) \leq A(\lambda_{n_o}) + \lambda_{n(r)} \exp (G(\lambda_{n(r)},\ \theta_*',\ \rho(\alpha,f))) + \\ + \frac{1}{1-\sqrt{r/R}}\ .$$

Now, if $\alpha_*(x) \in L^O$ and $\alpha_*(x) \not\in \Lambda$, from (3.3.8), (3.3.9) and (3.3.11), we obtain

$$\log M(r,f) \leq \log \frac{2}{\log (R/r)} + (1 + o(1)) \log \frac{2}{\log (R/r)}$$

$$+ G((2/\log (R/r))^{1+o(1)},\ \theta_*',\ \rho(\alpha,f)).$$

In view of Lemma 3.3.1, the above relation gives that

$$(1+o(1)) \log M(r,f) \leq G((2/\log (R/r))^{1+o(1)},\ \theta_*',\ \rho(\alpha,f)).$$

Since $\alpha_*(x) \in L^O$, this gives that $T(\alpha,\alpha_*,f) \leq \theta_*' = \theta_* + \varepsilon$ and so, letting $\varepsilon \to 0$, we have

(3.3.12) $\quad T(\alpha,\alpha_*,f) \le \theta_*.$

Obviously, (3.3.12) holds when $\theta_* = \infty$.

For the case $\alpha_*(x) \in \Lambda$, on using (3.3.8), (3.3.9) and (3.3.11), we get

(3.3.13) $\quad \log M(r,f) \le 5\ G((2/\log (R/r))^2;\ \theta_*';\ \rho(\alpha;f))$

for all r sufficiently near to R. Now, (3.3.12) follows easily from (3.3.13) for the case $\alpha_*(x) \in \Lambda$ also.

In view of (3.3.5) and (3.3.12) the proof of the theorem is complete.

REMARK. With $\alpha(x) = \log x$ and $\alpha_*(x) = x$, a result of Kapoor an Gopal [54] follows from the above theorem.

LEMMA 3.3.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,f)$, $1 < \rho(\alpha,f) < \infty$, and let $\alpha_*(x) \in$ Then

(3.3.14)
$$\liminf_{r \to R} \frac{\alpha_*(\log M(r,f))}{(\alpha_*(\log (R/(R-r))))^{\rho(\alpha,f)}} \ge \liminf_{k \to \infty} \frac{\alpha_*(\log |a_{n_k}|\ R^{\lambda_{n_k}})}{(\alpha_*(\log \lambda_{n_{k+1}}))^{\rho(\alpha,f)}}$$

for any increasing sequence $\{n_k\}$ of positive integers.

PROOF. Let the limit inferior on the right hand side of (3.3.14) be denoted by θ_*. Clearly $0 \le \theta_* \le \infty$. First, let $0 < \theta_* < \infty$. Then, given $\varepsilon > 0$, $\theta_* > \varepsilon$, there exists $k_o = k_o(\varepsilon)$ such that, for all $k > k_o$, we have

(3.3.15) $\log |a_{n_k}| R^{\lambda_{n_k}} > G(\lambda_{n_{k+1}}, \bar{\theta}_*, \rho(\alpha,f))$

where $G(\lambda_{n_{k+1}}, \bar{\theta}_*, \rho(\alpha,f)) = \alpha_*^{-1}(\bar{\theta}_*(\alpha_*(\log \lambda_{n_{k+1}}))^{\rho(\alpha,f)})$ and $\bar{\theta}_* = \theta_* - \varepsilon$. Set $r_k = R(1-1/\lambda_{n_k})$. Then, for $r_k \le r \le r_{k+1}$, using (3.3.15) and Cauchy's inequality, we obtain

$$\begin{aligned}\log M(r,f) &\ge \log |a_{n_k}| R^{\lambda_{n_k}} + \lambda_{n_k} \log (r/R)\\ &\ge G(\lambda_{n_{k+1}}, \bar{\theta}_*, \rho(\alpha,f)) + \lambda_{n_k} \log (r_k/R)\\ &= G(R/(R-r_{k+1}), \bar{\theta}_*, \rho(\alpha,f)) + \lambda_{n_k} \log (1-1/\lambda_{n_k})\\ &\ge G(R/(R-r), \bar{\theta}_*, \rho(\alpha,f)) + O(1).\end{aligned}$$

Since $\alpha_*(x) \in L^0$ and $\varepsilon > 0$ is arbitrary, (3.3.14) follows from the above inequality for the case $0 < \theta_* < \infty$. Obviously, (3.3.1· holds for $\theta_* = 0$. For $\theta_* = \infty$, the above arguments, with an arbitrarily large number in place of $\bar{\theta}_*$ give that the limit inferior on the left hand side of (3.3.14) is also ∞. This proves the lemma.

LEMMA 3.3.3. *Let the function $f(z)$, analytic in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,f)$, $1 < \rho(\alpha,f) < \infty$, (α,α_*)-logarithmic type $T(\alpha,\alpha_*,f)$ and lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$. Then $T(\alpha,\alpha_*,f) = \Phi_*$ and $t(\alpha,\alpha_*,f) = \phi_*$, where*

$$\begin{matrix}\Phi_*\\ \phi_*\end{matrix} = \lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha_*(\log \mu(r))}{(\alpha_*(\log (R/(R-r))))^{\rho(\alpha,f)}}.$$

PROOF. Since $\rho(\alpha,f) > 1$, by the proof of Lemma 3.2.1, it follows that $\mu(r)$ is an unbounded function of r. Now, by (3.2.6), we hav

$$(3.3.16) \quad \log M(r,f) \le \log \mu(r) + \log\log \mu(r+2\tfrac{R-r}{3}) + 2\log \tfrac{R}{R-r} + O$$

$$\le (1+o(1)) \log \mu(r+2\tfrac{R-r}{3}) + 2\log\tfrac{R}{R-r} + O(1)$$

Now, let $\Phi_* < \infty$. Then, for $r > r_0 = r_0(\varepsilon)$, $\varepsilon > 0$, we have

$$\log \mu(r + 2\tfrac{R-r}{3}) \le G(\tfrac{3R}{R-r}, \Phi_*', \rho(\alpha,f))$$

where $\Phi_*' = \Phi_* + \varepsilon$. Thus, by (3.3.16), we obtain

$$\log M(r,f) \le (1+o(1))\ G(\tfrac{3R}{R-r}, \Phi_*', \rho(\alpha,f)) + 2\log\tfrac{R}{R-r} + O(1)$$

$$= (1+o(1))\ G(\tfrac{3R}{R-r}, \Phi_*', \rho(\alpha,f)),$$

in view of Lemma 3.3.1. Since $\alpha_*(x) \in L^0$, the above relation gives that $T(\alpha,\alpha_*,f) \le \Phi_*' = \Phi_* + \varepsilon$ and so, letting $\varepsilon \to 0$, we get

$$(3.3.17) \qquad T(\alpha,\alpha_*,f) \le \Phi_* .$$

Obviously, (3.3.17) holds for $\Phi_* = \infty$.

Similarly, it can be shown that

$$(3.3.18) \qquad t(\alpha,\alpha_*,f) \le \phi_* .$$

The lemma now follows easily from (3.3.17) and (3.3.18) in view of the well known relation $\mu(r) \le M(r,f)$.

LEMMA 3.3.4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,f)$, $1 < \rho(\alpha,f) < \infty$, and lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$. Assume that for all c, $0 < c < \infty$, and all d, $1 < d < \infty$,

$$(3.3.19) \qquad \frac{dG(x,c,d)}{dx} \le G(x,c,d)/x \quad \text{for } x > x_0(c,d)$$

and

(3.3.20) $$x^2 \frac{dG(x,c,d)}{dx} \to \infty \text{ as } x \to \infty$$

hold. Further, if $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *is a nondecreasin function of* n *for* $n > n_o$, *then*

(3.3.21) $$t(\alpha,\alpha_*,f) \leq \liminf_{n\to\infty} \frac{\alpha_*(\log|a_n| R^{\lambda_n})}{(\alpha_*(\log\lambda_n))^{\rho(\alpha,f)}} .$$

PROOF. Since $\Psi(n)$ is ultimately nondecreasing and α-logarithmic order $\rho(\alpha,f)$ of $f(z)$ satisfies $\rho(\alpha,f) > 1$, by the proof of Theorem 3.2.2, we have $\psi(n) > \psi(n-1)$ for infinitely many values of n. Further, $\psi(n) \to R$ as $n \to \infty$. When $\psi(n) > \psi(n-1)$ the term $a_n z^{\lambda_n}$ becomes the maximum term and we have

$$\mu(r) = |a_n| r^{\lambda_n} \quad \text{for } \psi(n-1) \leq r < \psi(n).$$

Now, first let $0 < t(\alpha,\alpha_*,f) < \infty$. Then, by Lemma 3.3.3, given $\varepsilon > 0$ there exists $r_o = r_o(\varepsilon)$ such that, for $r > r_o$, we have

$$\log \mu(r) > G(R/(R-r), \bar{t}, \rho(\alpha,f))$$

where $G(R/(R-r),\bar{t}, \rho(\alpha,f)) = \alpha_*^{-1}(\bar{t}(\alpha_*(\log(R/(R-r))))^{\rho(\alpha,f)})$ and $\bar{t} = t(\alpha,\alpha_*,f)-\varepsilon > 0$. Let $a_{n_1} z^{\lambda_{n_1}}$ and $a_{n_2} z^{\lambda_{n_2}}$ ($n_1 > n_o$ and $\psi(n_1-1) > r_o$) be two consecutive maximum terms so that $n_1 \leq n_2-1$. Then

$$\log|a_{n_2}| + \lambda_{n_2} \log r > G(R/(R-r), \bar{t}, \rho(\alpha,f))$$

for every r satisfying $\psi(n_2-1) \le r < \Psi(n_2)$. Let $n_1 \le n \le n_2-1$. $\psi(n_1) = \Psi(n_1+1) = \ldots = \Psi(n) = \ldots = \Psi(n_2-1)$ and

$$|a_n|\, r^{\lambda_n} = |a_{n_2}|\, r^{\lambda_{n_2}} \text{ for } r = \Psi(n).$$

Hence

$$(3.3.22) \quad \log |a_n| R^{\lambda_n} \ge \lambda_n \log (R/\Psi(n)) + G(R/(R-\Psi(n)), \bar{t}, \rho(\alpha, f))$$

$$\ge \lambda_n \frac{R-\Psi(n)}{R} + G(R/(R-\Psi(n)),\ \bar{t},\ \rho(\alpha, f)).$$

By the hypothesis of the theorem, we get $G(x, \bar{t}, \rho(\alpha, f)) < x$ and $dG(x, \bar{t}, \rho(\alpha, f))/dx \le G(x, \bar{t}, \rho(\alpha, f))/x$ for $x > x^o = x^o(\bar{t}, \rho(\alpha, f))$. Now, consider the function

$$H_*(x) = \lambda_n/x + G(x, \bar{t}, \rho(\alpha, f)).$$

For all sufficiently large values of n, let $x_*(n)$ be the point such that

$$H_*(x_*(n)) = \min_{x^o < x < \infty} H_*(x).$$

Using (3.3.19), (3.3.20) and the techniques used in the proof of Theorem 3.2.2, we get

$$(3.3.23) \qquad \lambda_n/(x_*(n))^2 = \left.\frac{dG(x, \bar{t}, \rho(\alpha, f))}{dx}\right|_{x=x_*(n)}$$

$$\le G(x_*(n), \bar{t}, \rho(\alpha, f))/x_*(n).$$

If $\alpha_*(x) \in \Lambda$, since $G(x, \bar{t}, \rho(\alpha, f)) < x$ for $x > x^o$, by (3.3.23), we have

$$(3.3.24) \qquad \sqrt{\lambda_n} < x_*(n).$$

If $\alpha_*(x) \in L^0$ and $\alpha_*(x) \notin \Lambda$, then (3.3.23), on using (3.1.4), gives that

(3.3.25) $$\log \lambda_n = (1 + o(1)) \log x_*(n).$$

From (3.3.22), (3.3.24), (3.3.25) and the definition of $x_*(n)$, we get (3.3.21) for the case $0 < t(\alpha,\alpha_*,f) < \infty$. Obviously, (3.3.21) holds for $t(\alpha,\alpha_*,f) = 0$. For $t(\alpha,\alpha_*,f) = \infty$, the above analysis with an arbitrarily large number in place of $\bar{t}$ gives that the limit inferior on the right hand side of (3.3.21) is also ∞.

This completes the proof of the lemma.

<u>REMARK</u>. In particular $\alpha_*(x) = \log_p x$, $p \geq 0$, satisfies the conditions (3.3.19) and (3.3.20).

Combining Lemmas 3.3.2 and 3.3.4 we obtain a coefficient characterization of lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$ for a subclass of functions $f(z)$, analytic in D_R. This is our following theorem.

<u>THEOREM</u> 3.3.2. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R < \infty$, <u>be of</u> α-<u>logarithmic order</u> $\rho(\alpha,f)$, $1 < \rho(\alpha,f) < \infty$, <u>and lower</u> (α,α_*) <u>logarithmic type</u> $t(\alpha,\alpha_*,f)$. <u>Assume that</u> (3.3.19) <u>and</u> (3.3.20) <u>are satisfied</u>. <u>Further, let</u> $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ <u>be ultimately a nondecreasing function of</u> n <u>and let</u> $\alpha_*(\log \lambda_n) \sim \alpha_*(\log \lambda_{n+1})$ <u>as</u> $n \to \infty$. <u>Then</u>

$$t(\alpha,\alpha_*,f) = \liminf_{n \to \infty} \frac{\alpha_*(\log |a_n| R^{\lambda_n})}{(\alpha_*(\log \lambda_{n+1}))^{\rho(\alpha,f)}}.$$

Our next theorem gives a coefficient characterization of lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$ which does not require t nondecreasing nature of the function $\Psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$

<u>THEOREM</u> 3.3.3. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, <u>analytic in</u> D_R, $0 < R <$ <u>be of</u> α-<u>logarithmic order</u> $\rho(\alpha,f)$, $1 < \rho(\alpha,f) < \infty$, <u>and lower</u> $(\alpha,\alpha$ <u>logarithmic type</u> $t(\alpha,\alpha_*,f)$. <u>Assume that</u> (3.3.19) <u>and</u> (3.3.20) <u>are satisfied</u>. <u>Further, let</u> $\alpha_*(\log \lambda_{n_m}) \sim \alpha_*(\log \lambda_{n_{m+1}})$ <u>as</u> $m \to \infty$, <u>where</u> $\{\lambda_{n_m}\}$ <u>is the sequence of the principal indices of</u> $f(z)$. <u>Then</u>

$$t(\alpha,\alpha_*,f) = \max_{\{n_k\}} \left\{\liminf_{k\to\infty} \frac{\alpha_*(\log |a_{n_k}| R^{\lambda_{n_k}})}{(\alpha_*(\log \lambda_{n_{k+1}}))^{\rho(\alpha,f)}}\right\},$$

<u>where maximum is taken over all increasing sequences</u> $\{n_k\}$ <u>of positive integers</u>.

The theorem can be proved by using Lemma 3.3.2 and Theorem 3.3.2 and adopting the lines of the proof of Theorem 3.2.4 and so we omit the proof.

<u>REMARK</u>. Some results in [37] follow from Theorems 3.3.2 and 3.3.3 with the choice $\alpha(x) = \log x$ and $\alpha_*(x) = x$.

We finally prove a decomposition theorem for a function $f(z)$, analytic in D_R, with (α,α_*)-logarithmic type $T(\alpha,\alpha_*,f)$ and lower (α,α_*)-logarithmic type $t(\alpha,\alpha_*,f)$ such that $t(\alpha,\alpha_*,f) <$ $T(\alpha,\alpha_*,f)$.

THEOREM 3.3.4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, *analytic in* D_R, $0 < R < \infty$ *be of* α-*logarithmic order* $\rho(\alpha, f)$, $1 < \rho(\alpha, f) < \infty$, (α, α_*)-*logarithmic type* $T(\alpha, \alpha_*, f)$ *and lower* (α, α_*)-*logarithmic type* $t(\alpha, \alpha_*, f)$ *with* $t(\alpha, \alpha_*, f) < T(\alpha, \alpha_*, f)$. *Let* $t(\alpha, \alpha_*, f) < u < T(\alpha, \alpha_*, f)$. *Then*

$$f(z) = g_u(z) + h_u(z)$$

where α-*logarithmic order of* $g_u(z)$ *is atmost* $\rho(\alpha, f)$ *and* (α, α_*)-*logarithmic type of* $g_u(z)$ *is atmost* u *if its* α-*logarithmic order is* $\rho(\alpha, f)$. *Further*

$$h_u(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}$$

satisfies

$$t(\alpha, \alpha_*, f) \geq u \liminf_{p \to \infty} \left(\frac{\alpha_*(\log \lambda_{n_p})}{\alpha_*(\log \lambda_{n_{p+1}})}\right)^{\rho(\alpha, f)}.$$

PROOF. Let $g_u(z) = \Sigma' a_n z^{\lambda_n}$, where Σ' denotes the summation over n for which

$$\log |a_n| R^{\lambda_n} \leq G(\lambda_n, u, \rho(\alpha, f)) \tag{3.3.26}$$

where $G(\lambda_n, u, \rho(\alpha, f)) = \alpha_*^{-1}(u\ (\alpha_*(\log \lambda_n))^{\rho(\alpha, f)})$. Then, $g_u(z)$ is analytic in D_R and is of α-logarithmic order atmost $\rho(\alpha, f)$. If $g_u(z)$ is of α-logarithmic order $\rho(\alpha, f)$, it follows from Theorem 3.3.1 that (α, α_*)-logarithmic type of $g_u(z)$ atmost u. We now set

$$h_u(z) = f(z) - g_u(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}.$$

Then, by (3.3.26), we get

$$\text{(3.3.27)} \qquad \log |a_{n_p}| R^{\lambda_{n_p}} > G(\lambda_{n_p}, u, \rho(\alpha,f)),$$

for $p = 0,1,2,\ldots$. Let $r_p = R(1-1/\lambda_{n_p})$. For $r_p \le r \le r_{p+1}$, using Cauchy's inequality and (3.3.27), we have

$$\begin{aligned}\log M(r,f) &= \log |a_{n_p}| + \lambda_{n_p} \log r \\ &\ge \log |a_{n_p}| R^{\lambda_{n_p}} + \lambda_{n_p} \log (r_p/R) \\ &\ge G(\lambda_{n_p}, u, \rho(\alpha,f)) + \lambda_{n_p} \log (1-1/\lambda_{n_p}) \\ &= (1+o(1))\, G(\lambda_{n_p}, u, \rho(\alpha,f))\end{aligned}$$

for all sufficiently large values of p. Thus

$$\frac{\alpha_*((1+o(1))\log M(r,f))}{(\alpha_*(\log(R/(R-r))))^{\rho(\alpha,f)}} \ge u \frac{(\alpha_*(\log \lambda_{n_p}))^{\rho(\alpha,f)}}{(\alpha_*(\log (R/(R-r_{p+1}))))^{\rho(\alpha,f)}}$$

$$= u\left(\frac{\alpha_*(\log \lambda_{n_p})}{\alpha_*(\log \lambda_{n_{p+1}})}\right)^{\rho(\alpha,f)},$$

for all r sufficiently near to R. Since $\alpha_*(x) \in L^0$, the theorem follows from the above relation.

<u>REMARK</u>. With $\alpha(x) = \log x$ and $\alpha_*(x) = x$, a result of Kapoor and Gopal [54] follows from the above theorem.

CHAPTER 4

GENERALIZED ORDERS AND APPROXIMATION OF HARMONIC FUNCTIONS REGULAR IN A FINITE HYPERBALL IN R^{p+2}

4.1. The harmonic functions in R^{p+2}, $p = 1,2,\ldots,$ are solutions of Laplace's equation

$$(4.1.1) \qquad \frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \ldots + \frac{\partial^2 H}{\partial x_{p+2}^2} = 0.$$

A polynomial of degree n in $x_1,x_2,\ldots,x_{p+2}$ is said to be a harmonic polynomial of degree n, if it satisfies (4.1.1).

Let $r,\theta_1,\theta_2,\ldots,\phi$ be the hyperspherical polar coordinates in (p+2)-dimensional Euclidean space R^{p+2}. Then,

$$\begin{aligned}
x_1 &= r \cos\theta_1 \\
x_2 &= r \sin\theta_1 \cos\theta_2 \\
x_3 &= r \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
x_p &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-1} \cos\theta_p \\
x_{p+1} &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-1} \sin\theta_p \cos\phi \\
x_{p+2} &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-1} \sin\theta_p \sin\phi
\end{aligned}$$

where $r \geq 0$, $0 \leq \theta_j \leq \pi (j = 1,2,\ldots,p)$ and $0 \leq \phi \leq 2\pi$.

In R^{p+2}, let H be a harmonic function with continuous second derivatives in some neighbourhood of the origin. Such a harmonic function H is said to be regular about origin and has the following hyperspherical harmonic expansion ([34, p. 85])

$$(4.1.2)\quad H \equiv H(r,\{\theta_k\},\phi) = \sum_{n=0}^{\infty} \sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_p=0}^{m_{p-1}} r^n\{A^1(n,\{m_k\})Y(n,\{m_k\},\{\theta_k\},\phi) + A^2(n,\{m_k\})Y(n,\{m_k\},\{\theta_k\},-\phi)\},$$

where $A^i(n,\{m_k\})$, $i = 1,2$, are constants,

$$(4.1.3)\quad Y(n,\{m_k\},\{\theta_k\},\pm\phi) = e^{\pm i m_p \phi} \prod_{k=1}^{p} (\sin\theta_k)^{m_k} C_{m_{k-1}-m_k}^{m_k+(p+1-k)/2}(\cos\theta_k),$$

$n = m_o \geq m_1 \geq m_2 \geq \cdots \geq m_p \geq 0$ and C_n^u are Gegenbauer polynomials defined in Section 1.12. It is known [24, pp. 237-240] that the functions $r^n\, Y(n,\{m_k\},\{\theta_k\},\pm\phi)$ are homogeneous harmonic polynomials of degree n. Further, for a given n, the number h(n,p) of linearly independent harmonic polynomials $r^n\, Y(n,\{m_k\},\{\theta_k\},\pm\phi)$ is

$$(4.1.4)\qquad h(n,p) = (2n+p)\frac{(n+p-1)!}{n!p!}.$$

It is also known [24, p. 239] that a homogeneous harmonic polynomial of degree n can be expressed as a linear combination of h(n,p) functions $r^n\, Y(n,\{m_k\},\{\theta_k\},\pm\phi)$ where Y's are defined by (4.1.3). Thus, a harmonic polynomial of degree j is given by (4.1.2) with $A^i(n,\{m_k\}) = 0$ for $n \geq j+1$, $i = 1,2$. We shall

denote the class of all harmonic polynomials of degree atmost j by π_j, $j = 0,1,2,\ldots$.

Let $S_r = \{(x_1,x_2,\ldots,x_{p+2}) : x_1^2 + x_2^2 + \ldots + x_{p+2}^2 = r^2\}$, $0 < r < \infty$, be the hypersphere of radius r centered at the origin and let $dS_r = r^{p+1} (\sin\theta_1)^p \ldots (\sin\theta_p)\, d\theta_1 \ldots d\theta_p\, d\phi$ denote surface area element on S_r. It is known [34 , p. 85] that the functions Y, given by (4.1.3), satisfy the following orthogonality property

(4.1.5)
$$\iint_{S_1} Y(m_o,\{m_k\}, \{\theta_k\}, \pm\phi)\, \overline{Y(n_o,\{n_k\}, \{\theta_k\}, \pm\phi)}\, dS_1 = 2\pi \prod_{k=1}^{p} \delta_{m_k}^{n_k} E_k(m_{k-1}, m_k)$$

where the integration is taken over the unit hypersphere S_1; $\delta_{m_k}^{n_k} = 1$, if $n_k = m_k$, $\delta_{m_k}^{n_k} = 0$ otherwise and

(4.1.6)
$$E_k(m_{k-1},m_k) = \frac{\pi 2^{k-2m_k-p}\, \Gamma(m_{k-1}+m_k+p+1-k)}{(m_{k-1}+\frac{p+1-k}{2})(m_{k-1}-m_k)!\, (\Gamma(m_k+\frac{p+1-k}{2}))^2}.$$

In R^{p+2}, let B_r, $0 < r \leq \infty$, denote the hyperball

$$B_r = \{(x_1,x_2,\ldots,x_{p+2}) : x_1^2 + x_2^2 + \ldots + x_{p+2}^2 < r^2\}$$

and let $\bar{B}_r$, $0 < r < \infty$,

$$\bar{B}_r = \{(x_1,x_2,\ldots,x_{p+2}) : x_1^2 + x_2^2 + \ldots + x_{p+2}^2 \leq r^2\},$$

denote the closure of B_r. We say that a harmonic function H is regular in the hyperball B_r if the series on the right hand side of (4.1.2) converges uniformly on compact subsets of B_r. Let H_R,

$0 < R \leq \infty$, denote the class of all harmonic functions H, regular in B_r for every $r \leq R$ but for no $r > R$. The functions $H \varepsilon H_\infty$ are called entire harmonic functions.

To study the growth of regular harmonic functions $H \varepsilon H_R$, $0 < R < \infty$, we introduce the following growth parameters.

DEFINITION 4.1.1. *A harmonic function $H \varepsilon H_R$, $0 < R < \infty$, is said to be of (α,β)-order $\rho(\alpha,\beta,H)$ and lower (α,β)-order $\lambda(\alpha,\beta,H)$, $0 \leq \lambda(\alpha,\beta,H) \leq \rho(\alpha,\beta,H) \leq \infty$, if*

$$\begin{matrix}\rho(\alpha,\beta,H) \\ \lambda(\alpha,\beta,H)\end{matrix} = \lim_{r \to R} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log M(r,H))}{\beta(R/(R-r))},$$

where $\alpha(x) \varepsilon \Lambda$ and $\beta(x) \varepsilon L^0$ satisfy either of the conditions (2.1.2) and (2.1.3) and

$$M(r,H) = \max_{\{\theta_k\},\phi} |H(r,\{\theta_k\},\phi)|.$$

REMARKS. (i) If (2.1.2) is satisfied, i.e., $\alpha(x) = \beta(x) = \log x$, then $\rho(\alpha,\beta,H)$ and $\lambda(\alpha,\beta,H)$, denoted by $\rho(H)$ and $\lambda(H)$, are called, respectively, *order* and *lower order* of H. A harmonic function $H \varepsilon H_R$, $0 < R < \infty$, is said to be of *slow growth* if $\rho(H) = 0$.

(ii) For the choices $\alpha(x) = \log_{q-1} x$, $q \geq 2$, and $\beta(x) = \log x$, $\rho(\alpha,\beta,H)$ and $\lambda(\alpha,\beta,H)$, denoted by $\rho_q(H)$ and $\lambda_q(H)$, are called, respectively, q-*order* and *lower* q-*order* of H. Clearly $\rho_2(H) = \rho(H)$ and $\lambda_2(H) = \lambda(H)$.

(iii) If q-order $\rho_q(H)$ of H satisfies $0 < \rho_q(H) < \infty$, then, with the choices $\alpha(x) = \log_{q-2} x$, $q \geq 3$, and $\beta(x) = x^{\rho_q(H)}$, $\rho(\alpha,\beta,H)$ and $\lambda(\alpha,\beta,H)$ give, respectively, q-*type* and *lower* q-*type*

of H. We note that the choice $\alpha(x) = x$ and $\beta(x) = x^d$, $0 < d < \infty$ is not permissible in Definition 4.1.1, in view of the condition (2.1.3). Thus, if the order $\rho(H)$ of H satisfies $0 < \rho(H) < \infty$, then we define the type $T(H)$ and lower type $t(H)$ of H as

$$(4.1.7) \qquad \begin{matrix} T(H) \\ t(H) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r,H)}{(R/(R-r))^{\rho(H)}} .$$

DEFINITION 4.1.2. A harmonic function $H \in H_R$, $0 < R < \infty$, having (α,β)-order $\rho(\alpha,\beta,H)$ and lower (α,β)-order $\lambda(\alpha,\beta,H)$ is said to be of regular (α,β)-growth if $\rho(\alpha,\beta,H) = \lambda(\alpha,\beta,H)$ and H is said to be of irregular (α,β)-growth if $\lambda(\alpha,\beta,H) < \rho(\alpha,\beta,H)$.

To study precisely the growth of harmonic functions of slow growth we introduce the concepts of α-logarithmic order and lower α-logarithmic order in the following Definition 4.1.3.

DEFINITION 4.1.3. A harmonic function $H \in H_R$, $0 < R < \infty$, is said to be of α-logarithmic order $\rho(\alpha,H)$ and lower α-logarithmic order $\lambda(\alpha,H)$, $0 \leq \lambda(\alpha,H) \leq \rho(\alpha,H) \leq \infty$, if

$$\begin{matrix} \rho(\alpha,H) \\ \lambda(\alpha,H) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,H))}{\alpha(\log(R/(R-r)))} ,$$

where $\alpha(x) \in \Lambda$ and (3.1.1) is satisfied.

In Section 4.2 we prove some lemmas that are used in the subsequent sections to study the growth and approximation of regular harmonic functions $H \in H_R$, $0 < R < \infty$. The characterizations of the growth parameters introduced in Definitions 4.1.1 and 4.1.3 and that of type and lower type, given by (4.1.7),

of a harmonic function $H \in H_R$, $0 < R < \infty$, are obtained in terms of the coefficients in its hyperspherical harmonic expansion (4.1.2), in Section 4.3. In Section 4.4, we have studied the influence of the growth of a harmonic function $H \in H_R$ $0 < R < \infty$, on its degree of approximation in L^δ-norm, $1 \leq \delta \leq \infty$. In Section 4.5, we obtain a necessary condition for a harmonic function $H \in H_R$, $0 < R < \infty$, to be of regular (α,β)-growth.

Throughout the present chapter, to avoid some trivial cases we shall assume that, for $H \in H_R$, $0 < R < \infty$, $M(r,H) \to \infty$ as $r \to R$.

4.2. In this section, we prove some lemmas that are needed in the sequel.

<u>LEMMA</u> 4.2.1. <u>Let the harmonic function</u> $H \in H_R$, $0 < R < \infty$, <u>be given by</u> (4.1.2). <u>Then, for any</u> $n \geq 0$ <u>and any</u> $r < R$, <u>we have</u>

$$\max_{i,\{m_k\}} \{|A^i(n,\{m_k\})| (\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\} \leq K(n+\tfrac{p}{2})^{p/2} \frac{M(r,H)}{r^n}$$

<u>where</u> K <u>is a constant and</u>

$$\bar{E}_k(m_{k-1},m_k) = \frac{2^{k-2m_k-p}\,\Gamma(m_{k-1}+m_k+p+1-k)}{(m_{k-1}-m_k)!\,(\Gamma(m_k+\frac{p+1-k}{2}))^2}. \tag{4.2.1}$$

<u>PROOF</u>. By the uniform convergence of the series (4.1.2) and the orthogonality property (4.1.5) of the functions Y's, given by (4.1.3), we have, for $r < R$,

$$2\pi\, r^{2n+p+1} A^i(n,\{m_k\}) \prod_{k=1}^{p} E_k(m_{k-1},m_k)$$
$$= \iint_{S_r} H(r,\{\theta_k\},\phi)\,(r^n\, \overline{Y(n,\{m_k\},\{\theta_k\},\pm\phi)})\, ds_r. \tag{4.2.2}$$

Using Schwartz's inequality and (4.1.5), (4.2.2) gives

$$2\pi r^n |A^i(n,\{m_k\})| \prod_{k=1}^{p} E_k(m_{k-1},m_k) \le M(r,H) \iint_{S_1} |Y(n,\{m_k\},\{\theta_k\},\pm\phi)| dS_1$$

$$\le M(r,H) \left(\iint_{S_1} |Y(n,\{m_k\},\{\theta_k\},\pm\phi)|^2 \, dS_1\right)^{1/2} \left(\iint_{S_1} dS_1\right)^{1/2}$$

$$= M(r,H) \left(2\pi \prod_{k=1}^{p} E_k(m_{k-1},m_k)\right)^{1/2} S_* ,$$

where S_*^2 is the surface area of the unit hypersphere S_1. Thus, we get

$$|A^i(n,\{m_k\})| \left(2\pi \prod_{k=1}^{p} E_k(m_{k-1},m_k)\right)^{1/2} \le S_* \, M(r,H)/r^n$$

or, using the definition of $\bar{E}_k(m_{k-1},m_k)$ in (4.2.1),

$$(4.2.3) \quad |A^i(n,\{m_k\})| \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2}$$

$$\le \frac{S_*}{\sqrt{2\pi}\,(\pi)^{p/2}} \left(\prod_{k=1}^{p} \left(m_{k-1} + \frac{p+1-k}{2}\right)\right)^{1/2} M(r,H)/r^n$$

$$\le \frac{S_*}{\sqrt{2\pi}\,(\pi)^{p/2}} (n + p/2)^{p/2} \, M(r,H)/r^n .$$

Since the right hand side of (4.2.3) is independent of $i, m_1, \ldots, m_p$, the lemma follows from (4.2.3) with $K = S_*/\sqrt{2\pi}(\pi)^{p/2}$.

LEMMA 4.2.2. ([94, Theorem 2.2.1]) Let C be a Jordan arc in the complex plane such that arc and chord on C are infinitesimals of the same order. Then, for any polynomial $P_n(z)$ of degree n, we have

$$\max_{z \in C} |P_n(z)| \le \tilde{K} \, n^{1/\delta} \left(\int_C |P_n(z)|^\delta \, |dz|\right)^{1/\delta}, \quad \delta > 0,$$

where $\tilde{K}$ is a constant depending on C and δ only.

LEMMA 4.2.3. *Let p, q, m, n and k be nonnegative integers such that $p \geq 1$, $k = 0, 1, \ldots, p-1$, $0 \leq q \leq m \leq n$, $n \geq 1$. Then*

$$\max_{-1 \leq x \leq 1} |(1-x^2)^{q/2} C_{m-q}^{q+(p-k)/2}(x)|$$

$$\leq K_o(4n+2p)^{p-k} \left(\frac{2^{1-2q-p+k} \Gamma(m+q+p-k)}{(\Gamma(q+\frac{p-k}{2}))^2 \Gamma(m-q+1)}\right)^{1/2},$$

where K_o is a constant.

PROOF. Set $\bar{P}_s(x) = (1-x^2)^{q/2+(s-1)/4} C_{m-q}^{q+(p-k)/2}(x)$, $s = 1, 2, \ldots, p-k$. Then $(\bar{P}_s(x))^4$ is a polynomial of degree $4m + 2(s-1)$. Applying Lemma 4.2.2, with $\delta = 1/4$, to $(\bar{P}_s(x))^4$, we obtain

$$\max_{-1 \leq x \leq 1} |\bar{P}_s(x)|^4 \leq K_1(4n+2(s-1))^4 \left(\int_{-1}^{1} |\bar{P}_s(x)|\, dx\right)^4,$$

since $m \leq n$. Here K_1 is a constant. Using Schwartz's inequality, the above relation gives that

$$\max_{-1 \leq x \leq 1} |\bar{P}_s(x)|^4 \leq K_1(4n+2(s-1))^4 \left(\int_{-1}^{1} |\bar{P}_{s+1}(x)|^2\, dx \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}\right)^2$$

and so

$$(4.2.4) \qquad \max_{-1 \leq x \leq 1} |\bar{P}_s(x)| \leq (4\pi^2 K_1)^{1/4}(4n+2p) \max_{-1 \leq x \leq 1} |\bar{P}_{s+1}(x)|,$$

since $s-1 \leq p-k-1 \leq p$. Writing (4.2.4) for $s = 1, 2, \ldots, p-k-1$ and multiplying the $p-k-1$ relations thus obtained, we have

$$(4.2.5) \qquad \max_{-1 \leq x \leq 1} |\bar{P}_1(x)|$$

$$\leq (4\pi^2 K_1)^{(p-k-1)/4}(4n+2p)^{p-k-1} \max_{-1 \leq x \leq 1} |\bar{P}_{p-k}(x)|.$$

Now, applying Lemma 4.2.2 to $(\bar{P}_{p-k}(x))^4$, we obtain

$$(4.2.6) \quad \max_{-1 \le x \le 1} |\bar{P}_{p-k}(x)|^4 \le K_2(4n+2p)^2 \left(\int_{-1}^{1} |\bar{P}_{p-k}(x)|^2\, dx\right)^2$$

$$= K_2(4n+2p)^2 \left(\int_{-1}^{1} (1-x^2)^{q+(p-k-1)/2} (C_{m-q}^{q+(p-k)/2}(x))^2 dx\right)^2$$

$$= K_2(4n+2p)^2 \left(\frac{\pi\, 2^{1-2q-p+k}\, \Gamma(m+q+p-k)}{(\Gamma(q+\frac{p-k}{2}))^2\, \Gamma(m-q+1)\, (m+\frac{p-k}{2})}\right)^2,$$

on using the following orthogonality property of Genenbauer polynomials (see [34 , p. 173])

$$(4.2.7) \quad \int_{-1}^{1} (1-x^2)^{u-1/2}\, C_n^u(x)\, C_m^u(x)\, dx = \frac{2^{1-2u}\, \pi\, \Gamma(n+2u)}{(\Gamma(u))^2\, \Gamma(n+1)(n+u)}\, \delta_m^n ,$$

where K_2 is a constant. From (4.2.5) and (4.2.6), we get

$$\max_{-1 \le x \le 1} |\bar{P}_1(x)| \le (4\pi^2 K_1)^{(p-k-1)/4}\, (K_2)^{1/4}\, (4n+2p)^{p-k} \times$$

$$\times \left(\frac{\pi\, 2^{1-2q-p+k}\, \Gamma(m+q+p-k)}{(\Gamma(q+\frac{p-k}{2}))^2\, \Gamma(m-q+1)\, (m+\frac{p-k}{2})}\right)^{1/2}.$$

The lemma now follows easily from the above inequality since $m + (p-k)/2 \ge 1/2$.

<u>LEMMA</u> 4.2.4. <u>The functions</u> Y's, <u>given by</u> (4.1.3), <u>satisfy</u>

$$\max_{\{\theta_k\},\phi} |Y(n,\{m_k\}, \{\theta_k\}, \pm\, \phi)| \le$$

$$K_*(4n+2p)^{p(p+1)/2} \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2},$$

<u>where</u> K_* <u>is a constant;</u> $\bar{E}_k(m_{k-1},m_k)$ <u>is given by</u> (4.2.1).

PROOF. By (4.1.3) and Lemma 4.2.3, we get

$$\max_{\{\theta_k\},\phi} |Y(n,\{m_k\}, \{\theta_k\}, \pm\phi)|$$

$$\leq \prod_{k=1}^{p} (\max_{\theta_k} |(\sin\theta_k)^{m_k} C_{m_{k-1}-m_k}^{m_k+(p+1-k)/2} (\cos\theta_k)|)$$

$$= \prod_{k=1}^{p} (\max_{-1\leq x\leq 1} |(1-x^2)^{m_k/2} C_{m_{k-1}-m_k}^{m_k+(p+1-k)/2} (x)|)$$

$$\leq \prod_{k=1}^{p} K_o(4n+2p)^{p+1-k} \left(\frac{2^{k-2m_k-p}\,\Gamma(m_k+m_{k-1}+p-k+1)}{(\Gamma(m_k+\frac{p-k+1}{2}))^2\ \Gamma(m_{k-1}-m_k+1)}\right)^{1/2}$$

$$\leq (K_o)^p\ (4n+2p)^{p^2-(1+2+\cdots+(p-1))} \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2}$$

$$= (K_o)^p\ (4n+2p)^{p^2-p(p-1)/2} \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2}.$$

The lemma now follows from the above inequality.

4.3. For a harmonic function $H \in H_R$, $0 < R < \infty$, the inter-relations between the growth parameters, introduced in Section 4.1, and the coefficients in the hyperspherical harmonic expansion (4.1.2) of H have been investigated in this section.

We first obtain a characterization of the harmonic functions in the class H_R, $0 < R \leq \infty$, in terms of their hyperspherical harmonic coefficients.

<u>THEOREM</u> 4.3.1. <u>Let</u> H, <u>given by</u> (4.1.2), <u>be a harmonic function in</u> R^{p+2}. <u>Then</u> $H \in H_R$, $0 < R \leq \infty$, <u>if and only if</u>,

$$\limsup_{n\to\infty} \left(\max_{i,\{m_k\}} \{|A^i(n,\{m_k\})| \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2}\}\right)^{1/n} = 1/R. \tag{4.3.1}$$

PROOF. First, let $H \in H_R$, $0 < R \leq \infty$, and denote the limit superior on the left hand side of (4.3.1) by $Z(H)$. By Lemma 4.2.1, for any $r < R$, we have

$$Z(H) \leq 1/r.$$

Since $r < R$ is arbitrary, the above relation gives that

$$Z(H) \leq 1/R. \tag{4.3.2}$$

For the case $R = \infty$, the necessity part of the theorem follows from (4.3.2). For the case $R < \infty$, if possible, let

$$Z(H) = 1/R_o < 1/R, \quad R_o > R. \tag{4.3.3}$$

Set

$$A_n(H) = \max_{i,\{m_k\}} \{|A^i(n,\{m_k\})| (\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\}. \tag{4.3.4}$$

Now, using (4.1.4) and Lemma 4.2.4, we obtain

$$\begin{aligned} (4.3.5)\quad &|\sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_p=0}^{m_{p-1}} (A^1(n,\{m_k\}) Y(n,\{m_k\},\{\theta_k\}, \phi) + \\ &\qquad A^2(n,\{m_k\})\ Y(n,\{m_k\},\{\theta_k\}, -\phi))| \\ &\leq K_*(4n+2p)^{p(p+1)/2} \sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \cdots \sum_{m_p=0}^{m_{p-1}} (|A^1(n,\{m_k\})| \\ &\qquad + |A^2(n,\{m_k\})|)\ (\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2} \\ &\leq K_*(4n+2p)^{p(p+1)/2}\ (2n+p)\ \frac{(n+p-1)!}{n!p!}\ A_n(H) \\ &\leq 2K_*(4n+2p)^{p(p+1)/2}\ (n+p)^p\ A_n(H), \end{aligned}$$

since, for $p \geq 1$,

$$(2n+p)\ \frac{(n+p-1)!}{n!p!} \leq (2n+p)(n+1)\ \ldots(n+p-1) \leq 2(n+p)^p. \tag{4.3.6}$$

Now, given any compact set in B_{R_o} we can choose $R' < R_o$ such that the compact set is contained in $\bar{B}_{R'}$. Set $R'' = (R'+R_o)/2$. Then, by (4.3.3), for $n \geq n(R'')$, we have

$$(4.3.7) \qquad A_n(H) \leq 1/(R'')^n .$$

From (4.3.5) we see that, in $\bar{B}_{R'}$, the hyperspherical harmonic expansion (4.1.2) of H is dominated by the series

$$(4.3.8) \qquad \sum_{n=0}^{\infty} 2K_*(4n+2p)^{p(p+1)/2} (n+p)^p A_n(H) (R')^n .$$

In view of (4.3.7), the series (4.3.8) is convergent. Hence the series (4.1.2) converges uniformly on compact subsets of B_{R_o} and so H is regular in B_{R_o}. But this is impossible, since $H \varepsilon H_R$ and $R_o > R$. Thus (4.3.3) can not hold and so, in view of (4.3.2) we get

$$Z(H) = 1/R$$

for the case $0 < R < \infty$ also. This proves the necessity part of the theorem.

Conversely, if (4.3.1) holds, then it follows from the proof of the necessity part of the theorem that the series (4.1.2) converges uniformly on compact subsets of B_R and so H is regular in B_R. This proves the sufficiency part of the theorem and the proof of the theorem is complete.

We now obtain coefficient characterizations of (α,β)-order and lower (α,β)-order of a harmonic function $H \varepsilon H_R$, $0 < R < \infty$.

THEOREM 4.3.2. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *given by* (4.1.2), *be of* (α,β)-*order* $\rho(\alpha,\beta,H)$. *Then, if* $\alpha(x) = \beta(x) = \log x$ *or* $\alpha \not\equiv \beta$, *we have*

$$(4.3.9) \qquad \rho(\alpha,\beta,H) + X(\alpha,\beta) = \tilde{P}(\theta_H)$$

where $X(\alpha,\beta) = 1$ *if* $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ *otherwise and*

$$\theta_H = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(n/\log^+(\max\limits_{i,\{m_k\}} \{|A^i(n,\{m_k\})|(\prod\limits_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\}R^n))}.$$

The equation (4.3.9) *continues to hold for* $\alpha \equiv \beta(\alpha(x) \not\equiv \log x)$ *provided* $\rho(\alpha,\alpha,H) \geq 1$.

PROOF. We define a function $h_o(z)$ as

$$(4.3.10) \qquad h_o(z) = \sum_{n=0}^{\infty} \frac{A_n(H)}{(n+p/2)^{p/2}} z^n,$$

where $A_n(H)$ are given by (4.3.4). By Theorem 4.3.1, $h_o(z)$ is analytic in D_R. Fora given b, $0 < b < 1$, using Lemma 4.2.1, we have

$$(4.3.11) \quad M(r,h_o) = \sum_{n=0}^{\infty} \frac{A_n(H)}{(n+p/2)^{p/2}} r^n$$

$$\leq K\, M(r+b(R-r),H) \sum_{n=0}^{\infty} \left(\frac{r}{r+b(R-r)}\right)^n$$

$$= \frac{K(r+b(R-r))}{b(R-r)} M(r+b(R-r),H)$$

$$\leq \frac{KR}{b(R-r)} M(r+b(R-r),H).$$

From (4.3.11), for all r sufficiently near to R, we have

$$(4.3.12)\quad \log M(r,h_o) \le \log\frac{K}{b} + \log\frac{R}{R-r} + \log M(r+b(R-r),H).$$
$$\le \max\{3\log M(r+b(R-r),H), 3\log\frac{R}{R-r}\}.$$

Thus, for $\alpha(x) \varepsilon \Lambda$, we obtain

$$(4.3.13)\quad \alpha(\log M(r,h_o)) \le \max\{\alpha(3\log M(r+b(R-r),H)), \alpha(3\log\frac{R}{R-r})\}$$

for all r sufficiently near to R. Hence, for $\beta(x) \varepsilon L^o$ and all r sufficiently near to R, we get

$$(4.3.14)\quad \frac{\alpha(\log M(r,h_o))}{\beta(R/(R-r))} \le \max\{\frac{\alpha(3\log M(r+b(R-r),H))}{\beta(R/((1-b)(R-r)))} \times$$
$$\times \frac{\beta(R/((1-b)(R-r)))}{\beta(R/(R-r))}, \frac{\alpha(3\log(R/(R-r)))}{\beta(R/(R-r))}\}.$$

If $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3), $\alpha \not\equiv \beta$, then $\alpha(x)/\beta(x) \to 0$ as $x \to \infty$, and so (4.3.14), on applying Lemma 2.2.1, gives that

$$(4.3.15)\qquad \rho(\alpha,\beta,h_o) \le \rho(\alpha,\beta,H).$$

From (4.3.14), it follows that (4.3.15) holds for the case $\alpha(x) = \beta(x) = \log x$ also, since $\log x/x \to 0$ as $x \to \infty$. Finally, for $\alpha(x) \varepsilon \Lambda$, we have

$$\limsup_{x\to\infty} \alpha(\log x)/\alpha(x) \le 1,$$

and so, since we have assumed that $\rho(\alpha,\alpha,H) \ge 1$ for the case $\alpha \equiv \beta$ ($\alpha(x) \ne \log x$), it follows from (4.3.14) that (4.3.15) holds in this case also.

Now, it follows, from Theorem 4.3.1, that the function $h^o(z)$, defined as below, is analytic in D_R,

$$(4.3.16)\qquad h^o(z) = \sum_{n=0}^{\infty} (4n+2p)^{p(p+1)/2} (n+p)^p A_n(H) z^n.$$

From (4.3.5) and (4.3.16) we now get

$$(4.3.17)\quad M(r,H) \leq 2\ K_* \sum_{n=0}^{\infty} (4n+2p)^{p(p+1)/2}(n+p)^p\ A_n(H)r^n$$

$$= 2\ K_*\ M(r,h^o),$$

which gives

$$\log M(r,H) - \log 2\ K_* \leq \log M(r,h^o)$$

and so for $\alpha(x)\ \varepsilon\ \Lambda$, since $M(r,H) \to \infty$ as $r \to R$, we have

$$(4.3.18)\quad \alpha((1+o(1))\log M(r,H)) \leq \alpha(\log M(r,h^o)),$$

as $r \to R$. Dividing (4.3.18) by $\beta(R/(R-r))$ and passing to limits we get

$$(4.3.19)\quad \rho(\alpha,\beta,H) \leq \rho(\alpha,\beta,h^o).$$

Now, let $f(z) = \sum_{n=0}^{\infty} a_n\ z^n$ be a function analytic in D_R, $0 < R < \infty$, with (α,β)-order $\rho(\alpha,\beta,f)$. Then ([123 , p. 265]), for any b, $0 < b < 1$ and $r < R$, we have

$$(4.3.20)\quad \frac{M(r,f) - |f(0)|}{R} \leq M(r,f') \leq \frac{M(r+b(R-r),f)}{b(R-r)},$$

where $f'(z)$ is the derivative of $f(z)$. Adopting the method used in deducing (4.3.15) from (4.3.11), we get, from (4.3.20), that

$$(4.3.21)_a\quad \rho(\alpha,\beta,f) = \rho(\alpha,\beta,f'),$$

if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$. Further, for the case $\alpha \equiv \beta(\alpha(x) \neq \log x)$ we have

$$(4.3.21)_b\quad \rho(\alpha,\beta,f) \leq \rho(\alpha,\beta,f') \leq \max(1,\rho(\alpha,\beta,f)).$$

We define

(4.3.22) $$f_{[k]}(z) = \sum_{n=1}^{\infty} a_n n^k z^n, \; k = 1,2,\ldots .$$

Since $f_{[1]}(z) = z f'(z)$, we obviously have $\rho(\alpha,\beta,f') = \rho(\alpha,\beta,f_{[1]})$. We also have, $f_{[k]}(z) = z f'_{[k-1]}(z)$, $k = 2,3,\ldots$. Thus, from $(4.3.21)_a$ and $(4.3.21)_b$, we obtain

$(4.3.23)_a$ $$\rho(\alpha,\beta,f) = \rho(\alpha,\beta,f_{[k]}), \; k = 1,2,3,\ldots$$

if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$; and if $\alpha \equiv \beta(\alpha(x) \neq \log x)$, we have

$(4.3.23)_b$ $$\rho(\alpha,\beta,f) \leq \rho(\alpha,\beta,f_{[k]}) \leq \max\,(1,\rho(\alpha,\beta,f)), k = 1,2,3,\ldots .$$

We now define a function $h(z)$ as

(4.3.24) $$h(z) = \sum_{n=0}^{\infty} A_n(H)\; z^n.$$

By Theorem 4.3.1, $h(z)$ is analytic in D_R. On using $(4.3.23)_a$ and $(4.3.23)_b$, we get

(4.3.25) $$\rho(\alpha,\beta,h_o) \leq \rho(\alpha,\beta,h) \leq \rho(\alpha,\beta,h^o) \leq P(\rho(\alpha,\beta,h_o)).$$

From (4.3.15), (4.3.19) and (4.3.25), since by the hypothesis of the theorem $\rho(\alpha,\alpha,H) \geq 1$, if $\alpha \equiv \beta(\alpha(x) \neq \log x)$, we have

(4.3.26) $$\rho(\alpha,\beta,H) = P(\rho(\alpha,\beta,h)).$$

Theorem now follows from (4.3.26), on applying (1.9.4) and Theorem 2.3.1 to the function $h(z)$ given by (4.3.24).

<u>THEOREM</u> 4.3.3. <u>Let the harmonic function</u> $H \in H_R$, $0 < R < \infty$, <u>given by</u> (4.1.2), <u>be of lower</u> (α,β)-<u>order</u> $\lambda(\alpha,\beta,H)$. <u>Then, if</u>

$\alpha \not\succ \beta$, *we have*

(4.3.27) $\quad \lambda(\alpha,\beta,H) + X(\alpha,\beta) = \max\limits_{\{n_j\}} [\tilde{P}(\theta_H(\{n_j\}))]$.

Here $X(\alpha,\beta) = 1$, *if* $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ *otherwise*,

$\theta_H(\{n_j\}) =$

$$\liminf_{j \to \infty} \frac{\alpha(n_{j-1})}{\beta(n_j/\log^+(\max\limits_{i,\{m_k\}} \{|A^i(n_j,\{m_k\})|(\prod\limits_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\}R^{n_j}))}$$

and the maximum in (4.3.27) *is taken over all increasing sequences* $\{n_j\}$ *of positive integers. For the case* $\alpha \equiv \beta(\alpha(x) \neq \log x)$, *equation* (4.3.27) *continues to hold provided* $\lambda(\alpha,\alpha,H) \geq 1$. *For the case* $\alpha(x) = \beta(x) = \log x$, (4.3.27) *holds provided the principal indices* $\{n_\ell\}$ *of the function*

$$h(z) = \sum_{n=0}^{\infty} (\max_{i,\{m_k\}} \{|A^i(n,\{m_k\})|(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\})z^n,$$

analytic in D_R, *satisfy the condition* $\log n_{\ell-1} \sim \log n_\ell$ *as* $\ell \to \infty$.

PROOF. Let $h_o(z)$ be the function defined by (4.3.10). If $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3), $\alpha \not\succ \beta$, then $\alpha(x)/\beta(x) \to 0$ as $x \to \infty$, and so, on applying Lemma 2.2.1, (4.3.14) gives

(4.3.28) $\quad \lambda(\alpha,\beta,h_o) \leq P(\lambda(\alpha,\beta,H))$.

By (4.3.14), it follows that (4.3.28) holds for the cases $\alpha \equiv \beta$ also.

On the other hand, (4.3.18) gives that

(4.3.29) $\lambda(\alpha,\beta,H) \leq \lambda(\alpha;\beta;h^{o})$

where the function $h^{o}(z)$; given by (4.3.16), is analytic in D_R.

Now, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D_R; $0 < R < \infty$; with lower $(\alpha;\beta)$-order $\lambda(\alpha,\beta,f)$. Set $f_{[k]}(z) = \sum_{n=1}^{\infty} a_n n^k z^n$; $k = 1,2,\ldots$ Adopting the method used in deducing (4.3.28) from (4.3.11), we get, from (4.3.20) that

(4.3.30) $\lambda(\alpha,\beta,f) \leq \lambda(\alpha,\beta,f') \leq P(\lambda(\alpha,\beta,f))$.

where $f'(z)$ is the derivative of $f(z)$. In view of $f_{[1]}(z) = z\, f'(z)$, we get $\lambda(\alpha,\beta,f') = \lambda(\alpha,\beta,f_{[1]})$. Thus, since $f_{[k]}(z) = z\, f'_{[k-1]}(z)$, $k = 2,3,\ldots,$ from (4.3.30), we get

(4.3.31) $\lambda(\alpha,\beta,f) \leq \lambda(\alpha,\beta,f_{[k]}) \leq P(\lambda(\alpha,\beta,f))$, $k = 1,2,3,\ldots$.

We now consider the function $h(z) = \sum_{n=0}^{\infty} A_n(H)\, z^n$, $A_n(H)$ are given by (4.3.4), analytic in D_R. On using (4.3.31) we get

(4.3.32) $\lambda(\alpha,\beta,h_o) \leq \lambda(\alpha,\beta,h) \leq \lambda(\alpha,\beta,h^{o}) \leq P(\lambda(\alpha,\beta,h_o))$.

Since, by the hypothesis of the theorem, $\lambda(\alpha,\alpha,H) \geq 1$ if $\alpha \equiv \beta(\alpha(x) \not\equiv \log x)$, from (4.3.28), (4.3.29) and (4.3.32), we get

(4.3.33) $\lambda(\alpha,\beta,H) = P(\lambda(\alpha,\beta,h))$.

Theorem now follows on applying (1.9.6) and Theorem 2.4.5 to the function $h(z) = \sum_{n=0}^{\infty} A_n(H) z^n$ and using (4.3.33).

REMARK. Let the harmonic function $H \in H_R$, $0 < R < \infty$, given by (4.1.2), be of lower (α,β)-order $\lambda(\alpha,\beta,H)$, where $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3). Assume that if $\alpha \equiv \beta$, we have $\lambda(\alpha,\alpha,H) \geq 1$. Then, using (4.3.33) and applying (2.4.34) to the function $h(z) = \sum_{n=0}^{\infty} A_n(H) z^n$, where $A_n(H)$ is defined by (4.3.4), we obtai another characterization, analogous to (4.3.27), of lower (α,β)-order $\lambda(\alpha,\beta,H)$ of the harmonic function H, in terms of the coefficients in the hyperspherical harmonic expansion (4.1.2) of H.

Since the choice $\alpha(x) = x$ and $\beta(x) = x^d$, $0 < d < \infty$, is not permissible in Definition 4.1.1, due to the condition (2.1.3), the coefficient characterizations obtained in Theorems 4.3.2 and 4.3.3 for a harmonic function $H \in H_R$, $0 < R < \infty$, are not applicak to the growth parameters given by (4.1.7). We now separately obta coefficient characterizations of type and lower type, given by (4.1.7), of a harmonic function. We first need the following lemma.

LEMMA 4.3.1. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *given by* (4.1.2), *be of order* $\rho(H)$ $(0 < \rho(H) < \infty)$, *type* $T(H)$ *and lower type* $t(H)$. *Then, the function* $h(z) = \sum_{n=0}^{\infty} A_n(H) z^n$, $A_n(H)$ *are given by* (4.3.4), *is analytic in* D_R *and is of order* $\rho(H)$, *type* $T(H)$ *and lower type* $t(H)$.

PROOF. Let $h_o(z)$ and $h^o(z)$ be given by (4.3.10) and (4.3.16), respectively. It follows from Theorem 4.3.1 that the functions $h_o(z)$, $h^o(z)$ and $h(z)$ are analytic in D_R. Further, by (4.3.15),

(4.3.19) and (4.3.25), it follows that each of the functions $h_o(z)$, $h^o(z)$ and $h(z)$ is of order $\rho(H)$. From (4.3.11), for any b, $0 < b < 1$, we have

$$\frac{\log M(r,h_o)}{(R/(R-r))^{\rho(H)}} \le \frac{\log \frac{K}{b} + \log \frac{R}{R-r}}{(R/(R-r))^{\rho(H)}} + \frac{\log M(r+b(R-r),H)}{(R/(R-r))^{\rho(H)}} .$$

If $T(h_o)$ and $t(h_o)$ are, respectively, type and lower type of the function $h_o(z)$, then, passing to limits, the above relation gives that

$$T(h_o) \le T(H)/(1-b)^{\rho(H)}, \; t(h_o) \le t(H)/(1-b)^{\rho(H)}$$

and since $b > 0$ is arbitrary, we have

(4.3.34) $$T(h_o) \le T(H), \; t(h_o) \le t(H).$$

On the other hand, by (4.3.17), we get

(4.3.35) $$T(H) \le T(h^o), \; t(H) \le t(h^o)$$

where $T(h^o)$ and $t(h^o)$ are, respectively, type and lower type of the function $h^o(z)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D_R, $0 < R < \infty$, with order $\rho(f)$ $(0 < \rho(f) < \infty)$, type $T(f)$ and lower type $t(f)$. If $f'(z)$ is the derivative of $f(z)$, then, by $(4.3.21)_a$, $f'(z)$ is of order $\rho(f)$. From (4.3.20), we now get $T(f) = T(f')$ and $t(f) = t(f')$, $T(f')$ and $t(f')$ being, respectively, the type and lower type of $f'(z)$. Let $f_{[k]}(z) = \sum_{n=1}^{\infty} a_n n^k z^n$, $k = 1,2,\ldots$. Then, by $(4.3.23)_a$, each $f_{[k]}(z)$ is of order $\rho(f)$. Let $T(f_{[k]})$ and

$t(f_{[k]})$, $k = 1,2,3,\ldots$, denote, respectively, the type and lower type of $f_{[k]}(z)$. Since $f_{[1]}(z) = z\, f'(z)$, we get $T(f_{[1]}) = T(f')$ and $t(f_{[1]}) = t(f')$ and so

(4.3.36) $\qquad T(f) = T(f_{[1]}),\ t(f) = t(f_{[1]})$.

From (4.3.36), in view of the relation $f_{[k]}(z) = z\, f'_{[k-1]}(z)$, $k = 2,3,\ldots$, we get

(4.3.37) $\qquad T(f) = T(f_{[k]}),\ t(f) = t(f_{[k]}),\ k = 1,2,\ldots$

Applying (4.3.37) to the functions $h_o(z)$, $h^o(z)$ and $h(z)$, we get

(4.3.38) $\qquad T(h) = T(h_o) = T(h^o),\ t(h) = t(h_o) = t(h^o)$,

where $T(h)$ and $t(h)$ are, respectively, type and lower type of the function $h(z) = \sum_{n=0}^{\infty} A_n(H)\, z^n$.

The lemma now follows from (4.3.34), (4.3.35) and (4.3.38).

We now have

<u>THEOREM</u> 4.3.4. <u>Let the harmonic function</u> $H \in H_R$, $0 < R < \infty$, <u>given by</u> (4.1.2) <u>be of order</u> $\rho(H)$ $(0 < \rho(H) < \infty)$ <u>and type</u> $T(H)$. <u>Then</u>

$$\frac{(\rho(H)+1)^{\rho(H)+1}}{\rho(H)^{\rho(H)}}\, T(H) = \nu_H$$

<u>where</u>

$$\nu_H = \limsup_{n\to\infty} \frac{\left(\log^+\left(\max_{i,\{m_k\}} \{|A^i(n,\{m_k\})|\left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2} R^n\}\right)\right)^{\rho(H)+1}}{n^{\rho(H)}}$$

PROOF. The theorem follows from Lemma 4.3.1 on applying (1.9.7) to the function $h(z) = \sum_{n=0}^{\infty} A_n(H)\, z^n$, where $A_n(H)$ are given by (4.3.4).

THEOREM 4.3.5. *Let the harmonic function* $H \in H_R$; $0 < R < \infty$, *given by* (4.1.2), *be of order* $\rho(H)$ $(0 < \rho(H) < \infty)$ *and lower type* $t(H)$. *Assume that the principal indices* $\{n_\ell\}$ *of the function*

$$h(z) = \sum_{n=0}^{\infty} \Big(\max_{i,\{m_k\}} \{ |A^i(n,\{m_k\})| \big(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k) \big)^{1/2} \} \Big)\, z^n,$$

analytic in D_R, *satisfy the condition that* $n_\ell \sim n_{\ell+1}$ *as* $\ell \to \infty$. *Then*

$$\frac{(\rho(H)+1)^{\rho(H)+1}}{\rho(H)^{\rho(H)}}\, t(H) = \max_{\{n_j\}} \Big\{ \liminf_{j \to \infty} (n_{j-1}) \times \left(\frac{\log^+\big(\max_{i,\{m_k\}} \{ |A^i(n_j,\{m_k\})| \big(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k) \big)^{1/2} \} R^{n_j} \big)}{n_j} \right)^{\rho(H)+1} \Big\}$$

where maximum is taken over all increasing sequences $\{n_j\}$ *of positive integers.*

PROOF. The theorem follows from Lemma 4.3.1 on applying (1.9.10) to the function $h(z) = \sum_{n=0}^{\infty} A_n(H)\, z^n$, $A_n(H)$ are given by (4.3.4).

Since the (α,β)-order $\rho(\alpha,\beta,H)$ of a harmonic function $H \in H_R$, $0 < R < \infty$, of slow growth satisfies $\rho(\alpha,\beta,H) = 0$, if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$ and $\rho(\alpha,\beta,H) \leq 1$ if $\alpha \equiv \beta(\alpha(x) \neq \log x)$, the coefficient characterizations obtained in Theorems 4.3.2 to 4.3.5 do not give any specific information about the influence of the growth of such harmonic functions on their hyperspherical harmonic coefficients. Thus, to study

precisely the influence of the growth of harmonic functions of zero order on their hyperspherical harmonic coefficients, we obtain coefficient characterizations of growth parameters introduced in Definition 4.1.3. We first need the following lemma.

LEMMA 4.3.2. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *given by* (4.1.2), *be of* α-*logarithmic order* $\rho(\alpha,H)$ *and lower* α-*logarithmic order* $\lambda(\alpha,H)$. *Assume that the function* $h(z) = \sum_{n=0}^{\infty} A_n(H)\, z^n$, $A_n(H)$ *given by* (4.3.4), *analytic in* D_R, *is of* α-*logarithmic order* $\rho(\alpha,h)$ *and lower* α-*logarithmic order* $\lambda(\alpha,h)$. *Then, if* $\rho(\alpha,H) \geq 1$, *we have*

$$\rho(\alpha,H) = \max\,(1,\rho(\alpha,h))$$

and, if $\lambda(\alpha,H) \geq 1$, *we also have*

$$\lambda(\alpha,H) = \max\,(1,\lambda(\alpha,h)).$$

PROOF. Let $h_o(z)$, analytic in D_R, be the function defined by (4.3.10). Then, from (4.3.13), with $b = 1/2$, for $\alpha(x) \in \Lambda$ and all r sufficiently near to R, we have

$$(4.3.39)\qquad \frac{\alpha(\log M(r,h_o))}{\alpha(\log (R/(R-r)))} \leq \max\,\left\{\frac{\alpha(3\,\log M(r+(R-r)/2,H))}{\alpha(\log (R/(R-r)))}\,,\ \frac{\alpha(3\,\log (R/(R-r)))}{\alpha(\log (R/(R-r)))}\right\}.$$

Since $\alpha(x) \in \Lambda$, we have $\alpha(3\,\log (R/(R-r))) \sim \alpha(\log(R/(R-r)))$ as $r \to R$. Thus, passing to limits in (4.3.39), we get

(4.3.40) $\rho(\alpha,h_o) \le \max(1,\rho(\alpha,H))$, $\lambda(\alpha,h_o) \le \max(1,\lambda(\alpha,H))$.

Now, let $h^o(z)$, analytic in D_R, be given by (4.3.16). From (4.3.18), for $\alpha(x) \varepsilon \Lambda$, we get

$$\frac{\alpha((1+o(1)) \log M(r,H))}{\alpha(\log (R/(R-r)))} \le \frac{\alpha(\log M(r,h^o))}{\alpha(\log (R/(R-r)))} .$$

The above relation, since $\alpha(x) \varepsilon \Lambda$, on passing to limits, gives that

(4.3.41) $\rho(\alpha,H) \le \rho(\alpha,h^o)$, $\lambda(\alpha,H) \le \lambda(\alpha,h^o)$.

Now, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D_R with α-logarithmic order $\rho(\alpha,f)$ and lower α-logarithmic order $\lambda(\alpha,f)$ and let $f_{[k]}(z) = \sum_{n=1}^{\infty} a_n n^k z^n$, $k = 1,2,\ldots$, be of α-logarithmic order $\rho(\alpha,f_{[k]})$ and lower α-logarithmic order $\lambda(\alpha,f_{[k]})$. From (4.3.20), on adopting the lines of arguments used in deducing (4.3.40) from (4.3.11), we get

$$\rho(\alpha,f) \le \rho(\alpha,f') \le \max(1,\rho(\alpha,f)),\ \lambda(\alpha,f) \le \lambda(\alpha,f') \le \max(1,\lambda(\alpha,f)),$$

where $\rho(\alpha,f')$ and $\lambda(\alpha,f')$ are, respectively, α-logarithmic order and lower α-logarithmic order of $f'(z)$, the derivative of $f(z)$. Hence

(4.3.42)
$$\rho(\alpha,f) \le \rho(\alpha,f_{[1]}) \le \max(1,\rho(\alpha,f)),$$
$$\lambda(\alpha,f) \le \lambda(\alpha,f_{[1]}) \le \max(1,\lambda(\alpha,f))$$

since, in view of $f_{[1]}(z) = z f'(z)$, we have $\rho(\alpha,f') = \rho(\alpha,f_{[1]})$ and $\lambda(\alpha,f') = \lambda(\alpha,f_{[1]})$. From (4.3.42), since $f_{[k]}(z) = z f'_{[k-1]}(z)$, $k = 2,3,\ldots$, we get

$$(4.3.43)\quad \begin{aligned} &\rho(\alpha,f) \le \rho(\alpha,f_{[k]}) \le \max\,(1,\rho(\alpha,f)),\\ &\lambda(\alpha,f) \le \lambda(\alpha,f_{[k]}) \le \max\,(1,\lambda(\alpha,f)),\ k = 1,2,3,\ldots . \end{aligned}$$

For the function $h(z) = \sum_{n=0}^{\infty} A_n(H)\, z^n$, analytic in D_R, from (4.3.43) we now have

$$(4.3.44)_a \quad \rho(\alpha,h_o) \le \rho(\alpha,h) \le \rho(\alpha,h^o) \le \max\,(1,\rho(\alpha,h_o)),$$

and

$$(4.3.44)_b \quad \lambda(\alpha,h_o) \le \lambda(\alpha,h) \le \lambda(\alpha,h^o) \le \max\,(1,\lambda(\alpha,h_o)).$$

The lemma now follows easily from (4.3.40), (4.3.41), $(4.3.44)_a$ and $(4.3.44)_b$.

We now have

THEOREM 4.3.6. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *given by* (4.1.2), *be of* α-*logarithmic order* $\rho(\alpha,H)$. *If* $\rho(\alpha,H) \ge 1$, *then*

$$\rho(\alpha,H) = \max\,(1,\theta_\alpha(H))$$

where

$$\theta_\alpha(H) = \limsup_{n\to\infty} \frac{\alpha(\log(\max\limits_{i,\{m_k\}} \{|A^i(n,\{m_k\})|\,(\prod\limits_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\}\, R^n))}{\alpha(\log n)} .$$

PROOF. In view of Lemma 4.3.2, the theorem now follows on applying (3.2.2) to the function $h(z) = \sum_{n=0}^{\infty} A_n(H) z^n$, $A_n(H)$ given by (4.3.4).

THEOREM 4.3.7. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *given by* (4.1.2), *be of lower* α-*logarithmic order* $\lambda(\alpha,H)$ *with*

$\lambda(\alpha,H) \geq 1$. *Assume that* (3.2.9) *and* (3.2.10) *are satisfied and that the principal indices* $\{n_\ell\}$ *of the function* $h(z) = \sum_{n=0}^{\infty} \left(\max_{i,\{m_k\}} \{|A^i(n,\{m_k\})|(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\}\right) z^n$, *analytic in* D_R, *satisfy* $\alpha(\log n_\ell) \sim \alpha(\log n_{\ell-1})$ *as* $\ell \to \infty$. *Then*

$$\lambda(\alpha,H) = \max(1, \theta_\alpha(H))$$

where

$$(4.3.45) \quad \theta_\alpha(H) = \max_{\{n_j\}} \{\liminf_{j\to\infty} \frac{\alpha(\log(\max_{i,\{m_k\}} \{|A^i(n_j,\{m_k\})|(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k))^{1/2}\} R^{n_j}))}{\alpha(\log n_{j+1})}\}$$

and the maximum in (4.3.45) *is taken over all increasing sequences* $\{n_j\}$ *of positive integers.*

PROOF. The theorem follows from Lemma 4.3.2 on applying Theorem 3.2.4 to the function $h(z) = \sum_{n=0}^{\infty} A_n(H) z^n$, where $A_n(H)$ are given by (4.3.4).

4.4. A harmonic function H, given by (4.1.2), is said to be regular on the closed hyperball $\bar{B}_R$, $0 < R < \infty$, if it is regular in some open hyperball $B_{R'}$, $R' > R$. Let $\bar{H}_{R_o}$, $0 < R_o < \infty$, denote the class of all harmonic functions H, regular on $\bar{B}_{R_o}$. Let w_1 and w_2 be two positive functions defined on $\bar{B}_{R_o}$, $0 < R_o < \infty$, such that $1/w_i$, $i = 1,2$, are bounded on $\bar{B}_{R_o}$. Further, w_1 is continuous and w_2 is integrable on $\bar{B}_{R_o}$. For $H \in \bar{H}_{R_o}$, set

$$(4.4.1) \quad ||H||_{R_o,\infty} = \max_{(x_1,x_2,\ldots,x_{p+2}) \in \bar{B}_{R_o}} [w_1(x_1,x_2,\ldots,x_{p+2}) \times |H(x_1,x_2,\ldots,x_{p+2})|]$$

and

(4.4.2) $||H||_{R_o,\delta} = (\iiint_{\bar{B}_{R_o}} w_2(r,\{\theta_k\},\phi)|H(r,\{\theta_k\},\phi)|^{\delta} dV)^{1/\delta}, 1 \leq \delta < \infty$

where $dV = r^{p+1}(\sin\theta_1)^p \ldots (\sin\theta_p)dr\, d\theta_1 \ldots d\theta_p\, d\phi$ is the volume element in R^{p+2}. Then $||\cdot||_{R_o,\infty}$ and $||\cdot||_{R_o,\delta}$ are called respectively, uniform norm and L^{δ}-norm on $\bar{H}_{R_o}$. In particular, when $w_1 \equiv 1$ on $\bar{B}_{R_o}$, we shall denote $||\cdot||_{R_o,\infty}$ by $||\cdot||^{*}_{R_o,\infty}$.

For $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, we set

(4.4.3) $$\Delta_{n,\infty}(H,R_o) = \inf_{h \in \pi_n} ||H-h||_{R_o,\infty}$$

and

(4.4.4) $$\Delta_{n,\delta}(H,R_o) = \inf_{h \in \pi_n} ||H-h||_{R_o,\delta}, \quad 1 \leq \delta < \infty,$$

where π_n consists of all harmonic polynomials of degree atmost n. Then $\Delta_{n,\infty}(H,R_o)$ and $\Delta_{n,\delta}(H,R_o)$ $(1 \leq \delta < \infty)$ are called errors in approximating the harmonic function $H \in \bar{H}_{R_o}$ by harmonic polynomials of degree atmost n, respectively, in uniform norm and L^{δ}-norm.

In this section we have studied the influence of the growth of a harmonic function $H \in H_R$, $0 < R < \infty$, as measured by the growth parameters introduced in Section 4.1, on the rates of decay of the approximation errors $\Delta_{n,\infty}(H,R_o)$ and $\Delta_{n,\delta}(H,R_o)$, $0 < R_o < R$, given by (4.4.3) and (4.4.4), respectively.

We need the following lemmas.

LEMMA 4.4.1. Let the harmonic function $H \in H_R, 0 < R \leq \infty$, and let $0 < R_o < R$. Then, for every q, $q = 0,1,2,\ldots$, there exist harmonic polynomials $\hat{h}_q \in \pi_q$ such that

$$(4.4.5) \quad ||H-\hat{h}_q||^*_{R_o,\infty} \leq K^* M(r,H)(4(q+1)+2p)^{p(p+1)/2}(q+p+1)^p \times$$
$$\times (q+p/2+1)^{p/2} (R_o/r)^{q+1}$$

for all r sufficiently near to R. Here K^* is a constant independent of q and r.

PROOF. Let the harmonic function $H \in H_R$ be given by (4.1.2) and let $\hat{h}_q$ denote the qth partial sum of the series (4.1.2) of H, i.e., $\hat{h}_q$ is given by (4.1.2) with $A^i(n,\{m_k\}) = 0$ for $n \geq q+1$. Clearly $\hat{h}_q \in \pi_q$. From (4.1.2) and (4.3.5), we now get

$$||H-\hat{h}_q||^*_{R_o,\infty} \leq 2 K_* \sum_{n=q+1}^{\infty} (4n+2p)^{p(p+1)/2}(n+p)^p A_n(H) R_o^n,$$

where $A_n(H)$ is given by (4.3.4). The above relation, in view of Lemma 4.2.1, gives that

$$(4.4.6) \quad ||H-\hat{h}_q||^*_{R_o,\infty} \leq 2K_* K M(r,H) \sum_{n=q+1}^{\infty} (4n+2p)^{p(p+1)/2}(n+p)^p \times$$
$$\times (n+p/2)^{p/2} (R_o/r)^n$$
$$= 2K_* KM(r,H)(4(q+1)+2p)^{p(p+1)/2}(q+1+p)^p(q+1+p/2)^{p/2} \times$$
$$\times (R_o/r)^{q+1} \sum_{n=0}^{\infty} (1 + \frac{4n}{4(q+1)+2p})^{p(p+1)/2}(1 + \frac{n}{q+1+p})^p \times$$
$$\times (1 + \frac{n}{q+1+p/2})^{p/2}(R_o/r)^n.$$

Now, for $q \geq 0$ and $r \geq r^*$, where $r^* = (R_o+R)/2$ if $R < \infty$ and $r^* = 2R_o$, if $R = \infty$, the last series in (4.4.6) is dominated by the series

$$\sum_{n=0}^{\infty}(1 + \frac{4n}{4+2p})^{p(p+1)/2}(1 + \frac{n}{p+1})^{p} (1 + \frac{n}{p/2+1})^{p/2} (R_o/r^*)^{n}.$$

Since the above series is convergent, the lemma follows from (4.4.6).

LEMMA 4.4.2. *Let the harmonic function* $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, *be given by* (4.1.2). *Then, for* $n \geq 1$, *we have*

$$(4.4.7) \qquad A_n(H)\ R_o^n \leq \bar{K}_\infty (n+p/2)^{p/2}\ \Delta_{n-1,\infty}(H,R_o)$$

and

$$(4.4.8) \quad A_n(H)R_o^n \leq \bar{K}_\delta (4n+2p)^{p(p+1)/2}(n+p/2)^p(n+p+2)\ \Delta_{n-1,\delta}(H,R_o),$$
$$1 \leq \delta < \infty,$$

where $A_n(H)$ *is given by* (4.3.4) *and* $\bar{K}_\infty$ *and* $\bar{K}_\delta$ *are constants independent of* n.

PROOF. From (4.2.2), for any $h \in \pi_{n-1}$, $n \geq 1$, and $r \leq R_o$, we have

$$(4.4.9) \quad 2\pi r^{2n+p+1}\ A^i(n,\{m_k\}) \prod_{k=1}^{p} E_k(m_{k-1},m_k)$$

$$= \iint_{S_r} (H(r,\{\theta_k\},\phi)-h(r,\{\theta_k\},\phi))(r^n\overline{Y(n,\{m_k\},\{\theta_k\},\pm\phi)})\,dS_r$$

where E_k's are defined by (4.1.6). Thus, using (4.4.9), with $r = R_o$, (4.1.5) and Schwartz's inequality, we obtain

$$2\pi R_o^n|A^i(n,\{m_k\})| \prod_{k=1}^{p} E_k(m_{k-1},m_k)$$

$$\leq ||H-h||^*_{R_o,\infty} \iint_{S_1} |Y(n,\{m_k\},\{\theta_k\},\ \pm\phi)|\ \ dS_1$$

$$\leq ||H-h||^*_{R_o,\infty}(\iint_{S_1} |Y(n,\{m_k\},\{\theta_k\},\pm\phi)|^2 dS_1)^{1/2}(\iint_{S_1} dS_1)^{1/2}$$

$$= ||H-h||^*_{R_o,\infty}\ (2\pi \prod_{k=1}^{p} E_k(m_{k-1},m_k))^{1/2}\ S_*\ ,$$

where S_*^2 is the surface area of the unit hypersphere S_1. The above relation easily gives that (see the proof of Lemma 4.2.1)

$$(4.4.10) \qquad A_n(H)R_o^n \leq \frac{S_*}{\sqrt{2\pi}(\pi)^{p/2}} (n+p/2)^{p/2} \, ||H-h||^*_{R_o,\infty}.$$

By the definition of the error $\Delta_{n,\infty}(H,R_o)$, for every positive integer n, there exists a harmonic polynomial $h_{n-1,\infty} \varepsilon \pi_{n-1}$ such that

$$(4.4.11) \qquad 2\,\Delta_{n-1,\infty}(H,R_o) \geq ||H-h_{n-1,\infty}||_{R_o,\infty}.$$

From the definition (4.4.1) of the norm $||\cdot||_{R_o,\infty}$, we now have

$$(4.4.12) \qquad ||H-h_{n-1,\infty}||_{R_o,\infty} \geq \frac{1}{J_1} ||H-h_{n-1,\infty}||^*_{R_o,\infty}$$

where $J_1 < \infty$ is such that $1/w_1 \leq J_1$ on $\bar{B}_{R_o}$. Taking, in particular, $h = h_{n-1,\infty}$ in (4.4.10) and using (4.4.11) and (4.4.12) we obtain

$$A_n(H)R_o^n \leq \frac{2S_*J_1}{\sqrt{2\pi}\,(\pi)^{p/2}} (n+p/2)^{p/2}\, \Delta_{n-1,\infty}(H,R_o).$$

This gives (4.4.7) with $\bar{K}_\infty = (2S_*J_1)/(\sqrt{2\pi}\,(\pi)^{p/2})$.

Now, from (4.4.9) and Lemma 4.2.4, for $r \leq R_o$ and any $h \,\varepsilon\, \pi_{n-1}$, $n \geq 1$, we have

$$2\pi r^{n+p+1}|A^i(n,\{m_k\})| \prod_{k=1}^{p} E_k(m_{k-1},m_k) \leq K_*(4n+2p)^{p(p+1)/2} \times$$

$$\times \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2} \iint_{S_r} |H(r,\{\theta_k\},\phi)-h(r,\{\theta_k\},\phi)| \, dS_r$$

where $\bar{E}_k$'s are given by (4.2.1). The above inequality gives that

$$r^{n+p+1}|A^i(n,\{m_k\})| \left(\prod_{k=1}^{p} \bar{E}_k(m_{k-1},m_k)\right)^{1/2} \leq K_* \frac{(4n+2p)^{p(p+1)/2}}{2\pi\,\pi^p} \times$$

$$\times \left(\prod_{k=1}^{p} \left(m_{k-1} + \frac{p+1-k}{2}\right)\right) \iint_{S_r} |H(r,\{\theta_k\},\phi)-h(r,\{\theta_k\},\phi)| \, dS_r.$$

or

$$A_n(H)r^{n+p+1} \leq K_* \frac{(4n+2p)^{p(p+1)/2}}{2\pi^{p+1}}(n+p/2)^p \iint_{S_r} |H(r,\{\theta_k\},\phi) - h(r,\{\theta_k\},\phi)|\, dS_r.$$

Multiplying both sides of the above inequality by dr and integrating from 0 to R_o, for any $h \in \pi_{n-1}$, $n \geq 1$, we get

$$\text{(4.4.13)} \quad \frac{R_o^{n+p+2}}{n+p+2} A_n(H) = A_n(H) \int_0^{R_o} r^{n+p+1}\, dr$$

$$\leq K_* \frac{(4n+2p)^{p(p+1)/2}}{2\pi^{p+1}}(n+p/2)^p \iiint_{\bar{B}_{R_o}} |H(r,\{\theta_k\},\phi) - h(r,\{\theta_k\},\phi)|\, dV.$$

By the definition (4.4.4) of the error $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta < \infty$, for every positive integer n there exists a harmonic polynomial $h_{n-1,\delta} \in \pi_{n-1}$ such that

$$\text{(4.4.14)} \qquad 2\,\Delta_{n-1,\delta}(H,R_o) \geq ||H-h_{n-1,\delta}||_{R_o,\delta}\,.$$

From the definition (4.4.2) of the norm $||\cdot||_{R_o,\delta}$, $1 \leq \delta < \infty$, we now get

$$\text{(4.4.15)} \quad ||H-h_{n-1,\delta}||_{R_o,\delta} \geq$$

$$\frac{1}{J_2^{1/\delta}}\Big(\iiint_{\bar{B}_{R_o}} |H(r,\{\theta_k\},\phi) - h_{n-1,\delta}(r,\{\theta_k\},\phi)|^\delta\, dV\Big)^{1/\delta}$$

where $J_2 < \infty$ is such that $1/w_2 \leq J_2$ on $\bar{B}_{R_o}$.

For $\delta = 1$, (4.4.8) now follows from (4.4.13), (4.4.14) and (4.4.15) with $\bar{K}_1 = J_2K_*/(\pi^{p+1} R_o^{p+2})$. Now, if $1 < \delta < \infty$, we choose $\eta > 0$ such that $1/\delta + 1/\eta = 1$. Then, by Holder's inequality we have

$$(4.4.16) \quad \iiint_{\bar{B}_{R_o}} |H(r,\{\theta_k\},\phi)-h(r,\{\theta_k\},\phi)|dV$$
$$\leq (\iiint_{\bar{B}_{R_o}} |H(r,\{\theta_k\},\phi) - h(r,\{\theta_k\},\phi)|^{\delta} dV)^{1/\delta}(\iiint_{\bar{B}_{R_o}} dV)^{1/\eta}.$$

From (4.4.13), (4.4.14), (4.4.15) and (4.4.16), for $1 < \delta < \infty$, we obtain

$$A_n(H)R_o^n \leq \frac{2J_2^{1/\delta}K_*}{2\pi^{p+1} R_o^{p+2}} (V(R_o))^{1/\eta}(4n+2p)^{p(p+1)/2}(n+p/2)^p(n+p+2) \times$$
$$\times \Delta_{n-1,\delta}(H,R_o),$$

where $V(R_o)$ is the volume of $\bar{B}_{R_o}$. This proves (4.4.8) for the case $1 < \delta < \infty$ also, with $\bar{K}_\delta = J_2^{1/\delta}K_*(V(R_o))^{1/\eta}/(\pi^{p+1} R_o^{p+2})$. The proof of the lemma is thus complete.

We now prove

THEOREM 4.4.1. *Let the harmonic function* $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$. *Then,* $H \in H_R$, $R_o < R$, *if and only if,*

$$(4.4.17) \qquad \limsup_{n\to\infty} (\Delta_{n,\delta}(H,R_o))^{1/n} = R_o/R, \quad 1 \leq \delta \leq \infty,$$

where the errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, *are given by* (4.4.3) *and* (4.4.4).

PROOF. First, let $H \in H_R$, $R > R_o$. We now choose the polynomials $\hat{h}_n \in \pi_n$, $n = 0,1,2,\ldots$, of Lemma 4.4.1 satisfying (4.4.5). Then,

using the definitions (4.4.3) and (4.4.4) of the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, we have

(4.4.18) $$\Delta_{n,\delta}(H,R_o) \leq \|H-\hat{h}_n\|_{R_o,\delta};\ 1 \leq \delta \leq \infty;\ n = 0,1,2,\ldots .$$

From the definitions (4.4.1) and (4.4.2) of the norms $\|\cdot\|_{R_o,\delta}$, $1 \leq \delta \leq \infty$, we now get

(4.4.19) $$\|H-\hat{h}_n\|_{R_o,\delta} \leq K_\delta^*\|H-\hat{h}_n\|^*_{R_o,\infty},\quad 1 \leq \delta \leq \infty,$$

where $K_\infty^* = \max\limits_{(x_1,x_2,\ldots,x_{p+2})\,\varepsilon\,\bar{B}_{R_o}} w_1(x_1,x_2,\ldots,x_{p+2})$ and $K_\delta^* = \left(\iiint\limits_{\bar{B}_{R_o}} w_2(r,\{\theta_k\},\phi)\,dV\right)^{1/\delta}$, $1 \leq \delta < \infty$, are finite since w_1 is continuous and w_2 is integrable on $\bar{B}_{R_o}$. By Lemma 4.4.1, we also have

(4.4.20) $$\|H-\hat{h}_n\|^*_{R_o,\infty} \leq K^*M(r,H)(4(n+1)+2p)^{p(p+1)/2}(n+p+1)^p \times \times(n+p/2+1)^{p/2}\ (R_o/r)^{n+1}$$

for all r sufficiently near to R. Combining (4.4.18), (4.4.19) and (4.4.20), for $1 \leq \delta \leq \infty$, we obtain

(4.4.21) $$\Delta_{n,\delta}(H,R_o) \leq K_\delta^*K^*M(r,H)(4(n+1)+2p)^{p(p+1)/2}(n+p+1)^p \times \times(n+p/2+1)^{p/2}\ (R_o/r)^{n+1}$$

for all r sufficiently near to R. The inequality (4.4.21), for $1 \leq \delta \leq \infty$, easily gives that

$$\limsup_{n\to\infty}\ (\Delta_{n,\delta}(H,R_o))^{1/n} \leq R_o/r$$

for all r sufficiently near to R and so

(4.4.22) $$\limsup_{n\to\infty} (\Delta_{n,\delta}(H,R_o))^{1/n} \le R_o/R.$$

On the other hand, using Lemma 4.4.2, we get

(4.4.23) $$R_o(\limsup_{n\to\infty} (A_n(H))^{1/n}) \le \limsup_{n\to\infty} (\Delta_{n,\delta}(H,R_o))^{1/n}, 1 \le \delta \le \infty,$$

where $A_n(H)$ is given by (4.3.4). In view of Theorem 4.3.1, since $H \in H_R$, the above relation gives that

(4.4.24) $$R_o/R \le \limsup_{n\to\infty} (\Delta_{n,\delta}(H,R_o))^{1/n}, 1 \le \delta \le \infty.$$

Combining (4.4.22) and (4.4.24), we obtain

$$\limsup_{n\to\infty} (\Delta_{n,\delta}(H,R_o))^{1/n} = R_o/R.$$

This proves the necessity part of the theorem.

Conversely, let (4.4.17) hold for $H \in \bar{H}_{R_o}$. Then, by (4.4.23), we have

(4.4.25) $$\limsup_{n\to\infty} (A_n(H))^{1/n} \le 1/R.$$

Now, if possible, let

(4.4.26) $$\limsup_{n\to\infty} (A_n(H))^{1/n} = 1/R' < 1/R, R' > R.$$

Then, by Theorem 4.3.1, H is regular in $B_{R'}$. Thus, (4.4.22), with R replaced by R′, gives

$$\limsup_{n\to\infty} (\Delta_{n,\delta}(H,R_o))^{1/n} \le R_o/R' < R_o/R,$$

for $1 \le \delta \le \infty$, which contradicts the supposition that (4.4.17)

holds for $H \in \bar{H}_{R_o}$. Hence (4.4.26) does not hold. The relation (4.4.25), now gives

$$\limsup_{n \to \infty} (A_n(H))^{1/n} = 1/R,$$

and so by Theorem 4.3.1, $H \in H_R$. This proves the sufficiency part of the theorem.

The theorem is thus proved.

We now obtain characterizations of (α,β)-order and lower (α,β)-order of a harmonic function $H \in H_R$, $0 < R < \infty$, in terms of its degree of approximation. We need the following lemma.

LEMMA 4.4.3. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *be of* (α,β)-*order* $\rho(\alpha,\beta,H)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,H)$. *Assume that* $0 < R_o < R$ *and* $1 \leq \delta \leq \infty$. *Then the function* $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, *where the approximation errors* $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, *are given by* (4.4.3) *and* (4.4.4), *is analytic in* D_R. *For*, $\alpha(x) = \beta(x) = \log x$ *or* $\alpha \not\equiv \beta$, *the* (α,β)-*order* $\rho(\alpha,\beta,h_\delta)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,h_\delta)$ *of* $h_\delta(z)$ *satisfy*

$$(4.4.27)_a \qquad \rho(\alpha,\beta,H) = P(\rho(\alpha,\beta,h_\delta)),$$

$$(4.4.27)_b \qquad \lambda(\alpha,\beta,H) = P(\lambda(\alpha,\beta,h_\delta)).$$

For the case $\alpha \equiv \beta(\alpha(x) \neq \log x)$, $(4.4.27)_a$ *continues to hold provided* $\rho(\alpha,\alpha,H) \geq 1$, *while* $(4.4.27)_b$ *continues to hold provided* $\lambda(\alpha,\alpha,H) \geq 1$.

PROOF. We define a function $h_\delta^o(z)$, $1 \leq \delta \leq \infty$, as

$$(4.4.28)\quad h_{\delta}^{o}(z) = \sum_{n=0}^{\infty} \frac{\Delta_{n,\delta}(H,R_o)}{(4(n+1)+2p)^{p(p+1)/2}\ (n+p+1)^{p}\ (n+p/2+1)^{p/2}} (z/R_o)^{n+1}.$$

By Theorem 4.4.1, $h_{\delta}^{o}(z)$ is analytic in D_R. Using (4.4.21), for a given b, $0 < b < 1$, and all r sufficiently near to R, we get

$$(4.4.29)\quad M(r,h_{\delta}^{o}) \le K_{\delta}^{*}K^{*}M(r+b(R-r),H) \sum_{n=0}^{\infty} \left(\frac{r}{r+b(R-r)}\right)^{n+1}$$

$$\le K_{\delta}^{*}K^{*} \frac{R}{b(R-r)} M(r+b(R-r),H).$$

Now, with (4.4.29) in place of (4.3.11) and proceeding as in the deduction of (4.3.14) from (4.3.11), we get

$$(4.4.30)\quad \frac{\alpha(\log M(r,h_{\delta}^{o}))}{\beta(R/(R-r))} \le \max\left\{\frac{\alpha(3\ \log M(r+b(R-r),H))}{\beta(R/(R-r))},\ \frac{\alpha(3\ \log (R/(R-r)))}{\beta(R/(R-r))}\right\}$$

for all r sufficiently near to R. If $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3) and $\alpha \not\equiv \beta$, then $\alpha(x)/\beta(x) \to 0$ as $x \to \infty$, and so, on using Lemma 2.2.1, (4.4.30) gives

$$(4.4.31)\quad \rho(\alpha,\beta,h_{\delta}^{o}) \le P(\rho(\alpha,\beta,H)),\ \lambda(\alpha,\beta,h_{\delta}^{o}) \le P(\lambda(\alpha,\beta,H))$$

where $\rho(\alpha,\beta,h_{\delta}^{o})$ and $\lambda(\alpha,\beta,h_{\delta}^{o})$ are, respectively, (α,β)-order and lower (α,β)-order of the function $h_{\delta}^{o}(z)$. From (4.4.30), it follows that (4.4.31) holds for the cases $\alpha \equiv \beta$ also.

We now define a function $h_{\delta}^{*}(z)$, $1 \le \delta \le \infty$, as

$$(4.4.32)\quad h_{\delta}^{*}(z) = \sum_{n=1}^{\infty} (4n+2p)^{p(p+1)}(n+p)^{p}(n+p/2)^{p}(n+p+2) \times$$

$$\times\ \Delta_{n-1,\delta}(H,R_o)(z/R_o)^{n}.$$

By Theorem 4.4.1, $h_\delta^*(z)$ is analytic in D_R. From (4.3.17) and Lemma 4.4.2 we now get

$$(4.4.33) \quad M(r,H) \leq 2K_*(A_o+\bar{K}_\delta \sum_{n=1}^{\infty} (4n+2p)^{p(p+1)}(n+p)^p(n+p/2)^p \times$$

$$\times (n+p+2)\Delta_{n-1,\delta}(H,R_o)(r/R_o)^n)$$

$$= 2K_*(A_o + \bar{K}_\delta M(r,h_\delta^*))$$

and so

$$(4.4.34) \qquad \rho(\alpha,\beta,H) \leq \rho(\alpha,\beta,h_\delta^*),\ \lambda(\alpha,\beta,H) \leq \lambda(\alpha,\beta,h_\delta^*),$$

where $\rho(\alpha,\beta,h_\delta^*)$ and $\lambda(\alpha,\beta,h_\delta^*)$ are, respectively, (α,β)-order and lower (α,β)-order of $h_\delta^*(z)$.

Now, consider the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$. By Theorem 4.4.1, $h_\delta(z)$ is analytic in D_R. In view of $(4.3.23)_a$ and $(4.3.23)_b$ we get

$$(4.4.35) \qquad \rho(\alpha,\beta,h_\delta^o) \leq \rho(\alpha,\beta,h_\delta) \leq \rho(\alpha,\beta,h_\delta^*) \leq P(\rho(\alpha,\beta,h_\delta^o)),$$

while on using (4.3.31) we get

$$(4.4.36) \quad \lambda(\alpha,\beta,h_\delta^o) \leq \lambda(\alpha,\beta,h_\delta) \leq \lambda(\alpha,\beta,h_\delta^*) \leq P(\lambda(\alpha,\beta,h_\delta^o)).$$

The lemma now follows from (4.4.31), (4.4.34), (4.4.35) and (4.4.36).

We now have

<u>THEOREM</u> 4.4.2. <u>Let the harmonic function</u> $H \in H_R$, $0 < R < \infty$, <u>be of</u> (α,β)-<u>order</u> $\rho(\alpha,\beta,H)$. <u>Assume that</u> $0 < R_o < R$ <u>and</u> $1 \leq \delta \leq \infty$. <u>Then, if</u> $\alpha(x) = \beta(x) = \log x$ <u>or</u> $\alpha \not\equiv \beta$, <u>we have</u>

$$(4.4.37) \qquad \rho(\alpha,\beta,H) + X(\alpha,\beta) = \tilde{P}(\theta_{H,\delta})$$

where $X(\alpha,\beta) = 1$ if $\alpha(x) = \beta(x) = \log x$; $X(\alpha,\beta) = 0$ otherwise and

$$\theta_{H,\delta} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(n/\log^+ \Delta_{n,\delta}(H,R_0)(R/R_0)^n)}; \ 1 \le \delta \le \infty,$$

where the approximation errors $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, are given by (4.4.3) and (4.4.4). The equation (4.4.37) continues to hold for $\alpha \equiv \beta(\alpha(x) \ne \log x)$ provided $\rho(\alpha,\alpha,H) \ge 1$.

PROOF. The theorem follows from Lemma 4.4.3, on applying (1.9.4) and Theorem 2.3.1 to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_0)(z/R_0)^n$, $1 \le \delta \le \infty$.

THEOREM 4.4.3. Let the harmonic function $H \in H_R$, $0 < R < \infty$, be of lower (α,β)-order $\lambda(\alpha,\beta,H)$. Assume that $0 < R_0 < R$ and $1 \le \delta \le \infty$. Then, if $\alpha \not\equiv \beta$, we have

$$(4.4.38) \qquad \lambda(\alpha,\beta,H) + X(\alpha,\beta) = \max_{\{n_k\}} [\tilde{P}(\theta_{H,\delta}(\{n_k\}))]$$

where $X(\alpha,\beta) = 1$ if $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ otherwise,

$$\theta_{H,\delta}(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(n_k/\log^+ \Delta_{n_k,\delta}(H,R_0)(R/R_0)^{n_k})},$$

the approximation errors $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, are given by (4.4.3) and (4.4.4) and the maximum in (4.4.38) is taken over all increasing sequences $\{n_k\}$ of positive integers. For the case $\alpha \equiv \beta(\alpha(x) \ne \log x)$ (4.4.38) holds provided $\lambda(\alpha,\alpha,H) \ge 1$. For $\alpha(x) = \beta(x) = \log x$, (4.4.38) holds provided the principal indices

$\{n_m\}$ <u>of the function</u> $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, <u>analytic in</u> D_R, <u>satisfy the condition that</u> $\log n_{m-1} \sim \log n_m$ <u>as</u> $m \to \infty$.

<u>PROOF</u>. The theorem follows from Lemma 4.4.3 on applying (1.9.6) and Theorem 2.4.5 to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$.

Since the choice $\alpha(x) = x$ and $\beta(x) = x^d$, $0 < d < \infty$, is not permissible in Definition 4.1.1, the results obtained in Theorems 4.4.2 and 4.4.3 for a harmonic function $H \in H_R$, $0 < R < \infty$, are not applicable to the growth parameters given by (4.1.7). We thus separately obtain the interrelations between the type and lower type, given by (4.1.7), of a harmonic function and its degree of approximation. We need the following lemma.

<u>LEMMA</u> 4.4.4. <u>Let the harmonic function</u> $H \in H_R$, $0 < R < \infty$, <u>be of order</u> $\rho(H)$ $(0 < \rho(H) < \infty)$, <u>type</u> $T(H)$ <u>and lower type</u> $t(H)$. <u>Assume that</u> $0 < R_o < R$ <u>and</u> $1 \leq \delta \leq \infty$. <u>Then the function</u> $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $\Delta_{n,\delta}(H,R_o)$ <u>being given by</u> (4.4.3) <u>and</u> (4.4.4), <u>is analytic in</u> D_R <u>and is of order</u> $\rho(H)$, <u>type</u> $T(H)$ <u>and lower type</u> $t(H)$.

<u>PROOF</u>. For $1 \leq \delta \leq \infty$, let the functions $h^o_\delta(z)$ and $h^*_\delta(z)$ be given by (4.4.28) and (4.4.32). By Theorem 4.4.1, the functions $h^o_\delta(z)$, $h^*_\delta(z)$ and $h_\delta(z)$ are analytic in D_R. It follows from (4.4.31), (4.4.34) and (4.4.35) that each of the functions $h_\delta(z)$, $h^o_\delta(z)$ and $h^*_\delta(z)$ is of order $\rho(H)$. From (4.4.29), for any b, $0 < b < 1$, and all r sufficiently near to R, we get

$$\frac{\log M(r,h_\delta^0)}{(R/(R-r))^{\rho(H)}} \le \frac{\log (K_\delta^* K^*/b)+\log \frac{R}{R-r}}{(R/(R-r))^{\rho(H)}} + \frac{\log M(r+b(R-r),H)}{(R/(R-r))^{\rho(H)}} .$$

The above relation easily gives that

$$T(h_\delta^0) \le T(H)/(1-b)^{\rho(H)}, \quad t(h_\delta^0) \le t(H)/(1-b)^{\rho(H)},$$

where $T(h_\delta^0)$ and $t(h_\delta^0)$ are, respectively, type and lower type of $h_\delta^0(z)$. Thus, since $b > 0$ is arbitrary, we obtain

(4.4.39) $\qquad T(h_\delta^0) \le T(H), \quad t(h_\delta^0) \le t(H).$

Now, by (4.4.33), we get

(4.4.40) $\qquad T(H) \le T(h_\delta^*), \quad t(H) \le t(h_\delta^*),$

where $T(h_\delta^*)$ and $t(h_\delta^*)$ are, respectively, type and lower type of $h_\delta^*(z)$.

We now consider the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_0)(z/R_0)^n$. Using (4.3.37), we get

(4.4.41) $\qquad T(h_\delta^0) = T(h_\delta) = T(h_\delta^*), \quad t(h_\delta^0) = t(h_\delta) = t(h_\delta^*).$

The lemma now follows from (4.4.39), (4.4.40) and (4.4.41).

We now have

THEOREM 4.4.4. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *be of order* $\rho(H)$, $0 < \rho(H) < \infty$, *and type* $T(H)$. *Assume that* $0 < R_0 < R$ *and* $1 \le \delta \le \infty$. *Then*

$$\frac{(\rho(H)+1)^{\rho(H)+1}}{\rho(H)^{\rho(H)}}\, T(H) = \limsup_{n\to\infty} \frac{(\log^+ \Delta_{n,\delta}(H,R_0)(R/R_0)^n)^{\rho(H)+1}}{n^{\rho(H)}},$$

where the approximation errors $\Delta_{n,\delta}(H,R_o)$; $1 \le \delta \le \infty$, *are given by* (4.4.3) *and* (4.4.4).

PROOF. The theorem follows from Lemma 4.4.4 on applying (1.9.7) to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \le \delta \le \infty$.

THEOREM 4.4.5. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *be of order* $\rho(H)$, $0 < \rho(H) < \infty$, *and lower type* $t(H)$. *Assume that* $0 < R_o < R$, $1 \le \delta \le \infty$, *and that the principal indices* $\{n_m\}$ *of the function* $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, *the approximation errors* $\Delta_{n,\delta}(H,R_o)$ *being given by* (4.4.3) *and* (4.4.4), *satisfy* $n_m \sim n_{m-1}$ *as* $m \to \infty$. *Then*

$$\frac{(\rho(H)+1)^{\rho(H)+1}}{\rho(H)^{\rho(H)}}\, t(H)$$

$$= \max_{\{n_k\}} \left\{\liminf_{k \to \infty} n_{k-1} \left(\frac{\log^+ \Delta_{n_k,\delta}(H,R_o)(R/R_o)^{n_k}}{n_k}\right)^{\rho(H)+1}\right\},$$

where the maximum is taken over all increasing sequences $\{n_k\}$ *of positive integers*.

PROOF. The theorem follows from Lemma 4.4.4 on applying (1.9.10) to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \le \delta \le \infty$.

Since the (α,β)-order $\rho(\alpha,\beta,H)$ of a harmonic function $H \in H_R$, $0 < R < \infty$, of slow growth satisfies $\rho(\alpha,\beta,H) = 0$ if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$ and $\rho(\alpha,\alpha,H) \le 1$ if $\alpha \equiv \beta(\alpha(x) \ne \log x)$, the results obtained in Theorems 4.4.2 to 4.4.5 do not give any specific information about the rate of decay of the approximation errors of such harmonic functions. Thus, to study the influence

of the growth of harmonic functions of zero order on their degree of approximation, we obtain characterizations of the growth parameters, given by Definition 4.1.3, of a harmonic function $H \in H_R$, $0 < R < \infty$, in terms of the approximation errors $\Delta_{n,\delta}(H,R_o)$, $0 < R_o < R$, $1 \leq \delta \leq \infty$, given by (4.4.3) and (4.4.4). We first need the following lemma.

LEMMA 4.4.5. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *be of* α-*logarithmic order* $\rho(\alpha,H)$ *and lower* α-*logarithmic order* $\lambda(\alpha,H)$. *Let* $0 < R_o < R$, $1 \leq \delta \leq \infty$ *and let the approximation errors* $\Delta_{n,\delta}(H,R_o)$ *be defined by* (4.4.3) *and* (4.4.4). *Assume that the function* $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, *analytic in* D_R, *is of* α-*logarithmic order* $\rho(\alpha,h_\delta)$ *and lower* α-*logarithmic order* $\lambda(\alpha,h_\delta)$. *Then, if* $\rho(\alpha,H) \geq 1$, *we have*

$$\rho(\alpha,H) = \max (1,\rho(\alpha,h_\delta))$$

and, if $\lambda(\alpha,H) \geq 1$, *we also have*

$$\lambda(\alpha,H) = \max (1,\lambda(\alpha,h_\delta)).$$

PROOF. Let the function $h_\delta^o(z)$, $1 \leq \delta \leq \infty$, analytic in D_R, be given by (4.4.28). Now, with (4.4.29) in place of (4.3.11) we proceed, as in the deduction of (4.3.39) from (4.3.11), to get

$$\frac{\alpha(\log M(r,h_\delta^o))}{\alpha(\log (R/(R-r)))} \leq \max \left\{\frac{\alpha(3 \log M(r+(R-r)/2,H))}{\alpha(\log (R/(R-r)))}, \frac{\alpha(3 \log (R/(R-r)))}{\alpha(\log (R/(R-r)))}\right\} \tag{4.4.42}$$

for all r sufficiently near to R and $\alpha(x) \in \Lambda$. Since $\alpha(x) \in \Lambda$,

we have $\alpha(3 \log (R/(R-r))) \sim \alpha(\log (R/(R-r)))$ as $r \to R$. Thus, on passing to limits in (4.4.42) we get

$$(4.4.43) \quad \rho(\alpha,h_\delta^0) \le \max (1,\rho(\alpha,H)),\ \lambda(\alpha,h_\delta^0) \le \max (1,\lambda(\alpha,H)).$$

Now, for the function $h_\delta^*(z)$, analytic in D_R and given by (4.4.32), using (4.4.33), we have

$$(4.4.44) \qquad \rho(\alpha,H) \le \rho(\alpha,h_\delta^*),\ \lambda(\alpha,H) \le \lambda(\alpha,h_\delta^*).$$

We now consider the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_0)(z/R_0)^n$, $1 \le \delta \le \infty$, where $\Delta_{n,\delta}(H,R_0)$ are given by (4.4.3) and (4.4.4). Applying (4.3.43) to $h_\delta^0(z)$, $h_\delta^*(z)$ and $h_\delta(z)$ we obtain

$$(4.4.45) \quad \begin{aligned} \rho(\alpha,h_\delta^0) &\le \rho(\alpha,h_\delta) \le \rho(\alpha,h_\delta^*) \le \max (1,\rho(\alpha,h_\delta^0)), \\ \lambda(\alpha,h_\delta^0) &\le \lambda(\alpha,h_\delta) \le \lambda(\alpha,h_\delta^*) \le \max (1,\lambda(\alpha,h_\delta^0)). \end{aligned}$$

The lemma now follows easily from (4.4.43), (4.4.44) and (4.4.45).

We now have

THEOREM 4.4.6. <u>Let the harmonic function</u> $H \in H_R$, $0 < R < \infty$, <u>be of</u> α-<u>logarithmic order</u> $\rho(\alpha,H)$ <u>with</u> $\rho(\alpha,H) \ge 1$. <u>Assume that</u> $0 < R_0 < R$ <u>and</u> $1 \le \delta \le \infty$. <u>Then</u>

$$\rho(\alpha,H) = \max \{1, \limsup_{n \to \infty} \frac{\alpha(\log \Delta_{n,\delta}(H,R_0)(R/R_0)^n)}{\alpha(\log n)}\},$$

<u>where the approximation errors</u> $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, <u>are given by</u> (4.4.3) <u>and</u> (4.4.4).

PROOF. In view of Lemma 4.4.5, the theorem follows on applying (3.2.2) to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$.

THEOREM 4.4.7. *Let the harmonic function $H \in H_R$, $0 < R < \infty$, be of lower α-logarithmic order $\lambda(\alpha,H)$ with $\lambda(\alpha,H) \geq 1$ and let (3.2.9) and (3.2.10) be satisfied. Assume that $0 < R_o < R$, $1 \leq \delta \leq \infty$, and $\Delta_{n,\delta}(H,R_o)/\Delta_{n+1,\delta}(H,R_o)$ is ultimately a non-decreasing function of n, where the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, are given by (4.4.3) and (4.4.4). Then*

$$\lambda(\alpha,H) = \max\{1, \liminf_{n \to \infty} \frac{\alpha(\log \Delta_{n,\delta}(H,R_o)(R/R_o)^n)}{\alpha(\log n)}\}.$$

PROOF. The theorem follows from Lemma 4.4.5 on applying Theorem 3.2.3 to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$.

THEOREM 4.4.8. *Let the harmonic function $H \in H_R$, $0 < R < \infty$, be of lower α-logarithmic order $\lambda(\alpha,H)$ with $\lambda(\alpha,H) \geq 1$ and let (3.2.9) and (3.2.10) be satisfied. Let $0 < R_o < R$, $1 \leq \delta \leq \infty$ and let the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, be given by (4.4.3) and (4.4.4). Assume that the principal indices $\{n_m\}$ of the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, analytic in D_R, satisfy $\alpha(\log n_m) \sim \alpha(\log n_{m-1})$ as $m \to \infty$. Then*

$$\lambda(\alpha,H) = \max(1, \theta_{\alpha,\delta}(H))$$

where

$$(4.4.46) \qquad \theta_{\alpha,\delta}(H) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{\alpha(\log \Delta_{n_k,\delta}(H,R_o)(R/R_o)^{n_k})}{\alpha(\log n_{k+1})}\}$$

and the maximum in (4.4.46) is taken over all increasing sequences $\{n_k\}$ of positive integers.

PROOF. In view of Lemma 4.4.5, the theorem follows on applying Theorem 3.2.4 to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$.

4.5. It follows from the definitions (4.4.3) and (4.4.4) of the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, that, for a harmonic function $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, and for any δ, $1 \leq \delta \leq \infty$, $\Delta_{n,\delta}(H,R_o)$ is a nonincreasing function of n. Let $\{n_q(\delta)\}_{q=0}^{\infty}$, $1 \leq \delta \leq \infty$, $n_o(\delta) = 0$, be the sequence of those values of n for which $\Delta_{n-1,\delta}(H,R_o) > \Delta_{n,\delta}(H,R_o)$, i.e.,

$$\text{(4.5.1)} \quad \Delta_{n,\delta}(H,R_o) = \Delta_{n_{q-1}(\delta),\delta}(H,R_o) \text{ for } n_{q-1}(\delta) \leq n < n_q(\delta), \quad q = 1,2,3,\ldots .$$

We now obtain a theorem which shows the influence of this sequence on the growth of a harmonic function $H \in H_R$, $R_o < R < \infty$.

THEOREM 4.5.1. Let the harmonic function $H \in H_R$, $0 < R < \infty$, be of (α,β)-order $\rho(\alpha,\beta,H)$ and lower (α,β)-order $\lambda(\alpha,\beta,H)$, where $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3). Assume that $0 < R_o < R$, $1 \leq \delta \leq \infty$, and that, for the case $\alpha \equiv \beta$, $\lambda(\alpha,\alpha,H) > 1$. Then the sequence $\{n_q(\delta)\}_{q=0}^{\infty}$, given by (4.5.1), satisfies

$$\lambda(\alpha,\beta,H) \leq \rho(\alpha,\beta,H) \liminf_{q \to \infty} \frac{\alpha(n_{q-1}(\delta))}{\alpha(n_q(\delta))} .$$

PROOF. Let $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$. Then, by Theorem 4.4.1, $h_\delta(z)$ is analytic in D_R and, since for the case $\alpha \equiv \beta$,

$\lambda(\alpha,\alpha,H) > 1$, it follows, by Lemma 4.4.3, that (α,β)-order and lower (α,β)-order of $h_\delta(z)$ are, respectively, $\rho(\alpha,\beta,H)$ and $\lambda(\alpha,\beta,H)$. We now define a function $\bar{h}_\delta(z)$ as

$$(4.5.2) \qquad \bar{h}_\delta(z) = (z/R_o)\, h_\delta(z) - h_\delta(z) + \Delta_{0,\delta}(H,R_o).$$

Then

$$\bar{h}_\delta(z) = \sum_{n=1}^{\infty} (\Delta_{n-1,\delta}(H,R_o) - \Delta_{n,\delta}(H,R_o))(z/R_o)^n$$

$$= \sum_{q=1}^{\infty} (\Delta_{n_{q-1}(\delta),\delta}(H,R_o) - \Delta_{n_q(\delta),\delta}(H,R_o))(z/R_o)^{n_q(\delta)}$$

$$= \sum_{q=1}^{\infty} \bar{a}_q\, z^{n_q(\delta)}$$

where $\bar{a}_q = (\Delta_{n_{q-1}(\delta),\delta}(H,R_o) - \Delta_{n_q(\delta),\delta}(H,R_o))/R_o^{n_q(\delta)}$. By (4.5.2) it is easily seen that $\bar{h}_\delta(z)$ is analytic in D_R and that (α,β)-order and lower (α,β)-order of $\bar{h}_\delta(z)$ are the same as those of $h_\delta(z)$. Hence $\bar{h}_\delta(z)$ is of (α,β)-order $\rho(\alpha,\beta,H)$ and lower (α,β)-order $\lambda(\alpha,\beta,H)$. Now applying Theorem 2.5.1 to the function $\bar{h}_\delta(z)$, we get

$$\lambda(\alpha,\beta,H) \le \rho(\alpha,\beta,H) \liminf_{q\to\infty} \frac{\alpha(n_{q-1}(\delta))}{\alpha(n_q(\delta))}.$$

This proves the theorem.

COROLLARY. *Let the harmonic function* $H \in H_R$, $0 < R < \infty$, *be of regular* (α,β)-*growth, where* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (2.1.3), *with* (α,β)-*order* $\rho(\alpha,\beta,H)$ *satisfying* $P(0) < \rho(\alpha,\beta,H) < \infty$. *Assume that* $0 < R_o < R$ *and* $1 \le \delta \le \infty$. *Then, the sequence* $\{n_q(\delta)\}$, *given by* (4.5.1), *satisfies*

$$\alpha(n_q(\delta)) \sim \alpha(n_{q+1}(\delta)) \quad \text{as} \quad q \to \infty.$$

REMARKS. (i) It is seen from the above corollary that for a harmonic function $H \in H_R$, $0 < R < \infty$, of regular (α,β)-growth the gaps in the sequence $\{n_q(\delta)\}$, defined by (4.5.1), are of the same order for any δ, $1 \leq \delta \leq \infty$.

(ii) If for a harmonic function $H \in H_R$, $0 < R < \infty$, the sequence $\{n_q(\delta)\}$, $1 \leq \delta \leq \infty$, given by (4.5.1), has wide gaps, i.e., $\liminf_{q\to\infty}(\alpha(n_q(\delta))/\alpha(n_{q+1}(\delta)) < 1$ then, by Theorem 4.5.1, H is of irregular (α,β)-growth.

CHAPTER 5

GENERALIZED ORDERS AND APPROXIMATION OF GASP AND GBSP REGULAR IN A FINITE DISC.

5.1. Generalized axisymmetric potentials (GASP's) are the solutions of the elliptic partial differential equation

$$(5.1.1) \qquad \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{2u}{y}\frac{\partial G}{\partial y} = 0, \ u > 0.$$

A polynomial of degree n in x and y is said to be a GASP polynomial of degree n if it satisfies (5.1.1). A GASP $G \equiv G_u$, regular about the origin has the following ultra-spherical harmonic expansion ([34, p. 174]),

$$(5.1.2) \qquad G(x,y) \equiv G(r,\theta) = \sum_{n=0}^{\infty} b_n \, r^n \, C_n^u(\cos\theta),$$

where $x = r\cos\theta$, $y = r\sin\theta$ and C_n^u are Gegenbauer polynomials.

Let D_R, as in the previous chapters, denote the open disc of radius R centered at the origin and let $\bar{D}_R$ denote the closure of D_R. A GASP G is said to be regular in the disc D_R if the series on the right hand side of (5.1.2) converges uniformly on compact subsets of D_R. In particular, a GASP G regular in D_∞ is called an entire GASP.

To study the growth of a GASP G, regular in a finite disc D_R, $0 < R < \infty$, we introduce the following growth parameters.

DEFINITION 5.1.1. *A GASP G, regular in a finite disc D_R, $0 < R < \infty$, is said to be of (α,β)-order $\rho(\alpha,\beta,G)$ and lower*

(α, β)-*order* $\lambda(\alpha,\beta,G)$, $0 \le \lambda(\alpha,\beta,G) \le \rho(\alpha,\beta,G) \le \infty$, *if*

$$\begin{matrix}\rho(\alpha,\beta,G) \\ \lambda(\alpha,\beta,G)\end{matrix} = \lim_{r \to R} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log M(r,G))}{\beta(R/(R-r))}$$

where $\alpha(x) \in \Lambda$ *and* $\beta(x) \in L^{o}$ *satisfy either of the conditions* (2.1.2) *and* (2.1.3) *and*

$$M(r,G) = \max_{0 \le \theta \le 2\pi} |G(r,\theta)|.$$

REMARK. If (2.1.2) is satisfied, i.e., $\alpha(x) = \beta(x) = \log x$, then $\rho(\alpha,\beta,G)$ and $\lambda(\alpha,\beta,G)$, denoted by $\rho(G)$ and $\lambda(G)$, are called, respectively, *order* and *lower order* of G. A GASP G, regular in D_R, $0 < R < \infty$, is said to be of *slow growth*, if $\rho(G) = 0$.

To study precisely the growth of a GASP G, regular in D_R, $0 < R < \infty$, of slow growth, we introduce some new growth parameters in the following Definition 5.1.2.

DEFINITION 5.1.2. *A* GASP G, *regular in a finite disc* D_R, $0 < R < \infty$, *is said to be of* α-*logarithmic order* $\rho(\alpha,G)$ *and lower* α-*logarithmic order* $\lambda(\alpha,G)$, $0 \le \lambda(\alpha,G) \le \rho(\alpha,G) \le \infty$, *if*

$$\begin{matrix}\rho(\alpha,G) \\ \lambda(\alpha,G)\end{matrix} = \lim_{r \to R} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log M(r,G))}{\alpha(\log (R/(R-r)))},$$

where $\alpha(x) \in \Lambda$ *and* (3.1.1) *is satisfied*.

A GASP G, given by (5.1.2), is said to be regular on the closed disc $\bar{D}_R$, $0 < R < \infty$, if it is regular in some open disc $D_{R'}$, $R' > R$. Let $\bar{G}_{R_o}$, $0 < R_o < \infty$, be the class of all GASP's G regular on $\bar{D}_{R_o}$, $0 < R_o < \infty$. Let w_1 and w_2 be two positive functions defined on $\bar{D}_{R_o}$, $0 < R_o < \infty$, such that $1/w_i$, $i=1,2$, are bounded on $\bar{D}_{R_o}$.

Further, w_1 is continuous and w_2 is integrable on $\bar{D}_{R_o}$. For $G \varepsilon \bar{G}_{R_o}$, set

$$(5.1.3) \qquad ||G||_{R_o,\infty} = \max_{x^2+y^2 \leq R_o^2} w_1(x,y)|G(x,y)|,$$

$$(5.1.4) \qquad ||G||_{R_o,\delta} = \left(\iint_{\bar{D}_{R_o}} w_2(x,y)\ |G(x,y)|^{\delta}\ dx\ dy\right)^{1/\delta}, 1 \leq \delta < \infty.$$

Then $||\cdot||_{R_o,\infty}$ and $||\cdot||_{R_o,\delta}$ are, respectively, uniform norm and L^{δ}-norm on $\bar{G}_{R_o}$. In particular, when $w_1 \equiv 1$ on $\bar{D}_{R_o}$ we shall denote $||\cdot||_{R_o,\infty}$ by $||\cdot||^{*}_{R_o,\infty}$.

For $G \varepsilon \bar{G}_{R_o}$, $0 < R_o < \infty$, we set

$$(5.1.5) \qquad \Delta_{n,\infty}(G,R_o) = \inf_{g\ \varepsilon\ \pi_n^*} ||G-g||_{R_o,\infty},$$

$$(5.1.6) \qquad \Delta_{n,\delta}(G,R_o) = \inf_{g\ \varepsilon\ \pi_n^*} ||G-g||_{R_o,\delta}\ ,\ 1 \leq \delta < \infty,$$

where π_n^* consists of all GASP polynomials of degree atmost n. Then $\Delta_{n,\infty}(G,R_o)$ and $\Delta_{n,\delta}(G,R_o)$ are called errors in approximating the GASP $G \varepsilon \bar{G}_{R_o}$ by GASP polynomials of degree atmost n, respectively, in uniform norm and L^{δ}-norm, $1 \leq \delta < \infty$.

In Section 5.2, we have studied the influence of the growth of a GASP G, regular in the disc D_R, $0 < R < \infty$, as measured by the growth parameters given in Definitions 5.1.1 and 5.1.2, on the rates of decay of the approximation errors $\Delta_{n,\delta}(G,R_o)$, $0 < R_o < R$ and $1 \leq \delta \leq \infty$, given by (5.1.5) and (5.1.6). To this end, in this section, we have obtained the characterizations of

uniform convergence of the series (5.1.2) on compact subsets of D_R and the orthogonality property (4.2.7) of the Gegenbauer polynomials, for any $r < R$, we have

$$(5.2.2)\quad b_n r^n \frac{2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)} \left(\frac{\Gamma(u+1/2)}{\Gamma(2u)}\right)^2$$

$$= \int_0^\pi \sin^{2u}\theta\, C_n^u(\cos\theta)\ G(r,\theta)\ d\theta ,$$

where b_n's are the coefficients in the expansion (5.1.2) of G. On using Schwartz's inequality and (4.2.7), from (5.2.2), we get

$$|b_n| r^n \frac{2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)} \left(\frac{\Gamma(u+1/2)}{\Gamma(2u)}\right)^2 \leq M(r,G) \int_0^\pi \sin^{2u}\theta\, |C_n^n(\cos\theta)|\, d\theta$$

$$\leq M(r,g)\left(\int_0^\pi \sin^{2u}\theta\, |C_n^u(\cos\theta)|^2 d\theta\right)^{1/2}\left(\int_0^\pi \sin^{2u}\theta\, d\theta\right)^{1/2}$$

$$\leq M(r,G)\ \left(\frac{\pi 2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)}\right)^{1/2} \frac{\Gamma(u+1/2)}{\Gamma(2u)} .$$

This easily gives that

$$(5.2.3)\quad |b_n|\ r^n \leq M(r,G)\ \left(\frac{\pi(n+u)}{2^{2u-1}} \frac{\Gamma(n+1)}{\Gamma(n+2u)}\right)^{1/2} \frac{\Gamma(2u)}{\Gamma(u+1/2)} .$$

It is known [122, p.97] that

$$C_n^u(\cos\theta) = \frac{2^{1-2u}}{(\Gamma(u))^2} \frac{\Gamma(n+2u)}{n!} \int_0^\pi (\cos\theta + i\sin\theta\cos\phi)^n \sin^{2u-1}\phi\, d\phi .$$

Thus, for $0 \leq \theta \leq 2\pi$, we get

$$(5.2.4)\quad |C_n^u(\cos\theta)| \leq \frac{2^{1-2u}}{(\Gamma(u))^2} \frac{\Gamma(n+2u)}{n!} \int_0^\pi \sin^{2u-1}\phi\, d\phi$$

$$= \frac{\Gamma(n+2u)}{\Gamma(2u)\ \Gamma(n+1)} .$$

Now, let $\hat{g}_k$ denote the kth partial sum of the expansion (5.1.2) of the GASP G, i.e, $\hat{g}_k$ is given by (5.1.2) with $b_n = 0$ for $n \geq k+1$. Then, $\hat{g}_k \varepsilon \pi_k^*$. Using (5.1.2), (5.2.3) and (5.2.4), we get

$$(5.2.5) \qquad ||G-\hat{g}_k||^*_{R_o,\infty} \leq \sum_{n=k+1}^{\infty} |b_n| R_o^n |C_n^u(\cos\theta)|$$

$$\leq \frac{M(r,G)}{\Gamma(u+1/2)} \left(\frac{\pi}{2^{2u-1}}\right)^{1/2} \sum_{n=k+1}^{\infty} \left(\frac{(n+u)\ \Gamma(n+2u)}{\Gamma(n+1)}\right)^{1/2} (R_o/r)^n.$$

Since $\Gamma(x+a)/\Gamma(x) \sim x^a$ as $x \to \infty$, it follows that $(n+u)\ \Gamma(n+2u)/\Gamma(n+1) \sim n^{2u}$ as $n \to \infty$. Thus

$$(5.2.6) \qquad ((n+u)\ \Gamma(n+2u)/\Gamma(n+1))^{1/2} < 2n^u$$

for $n > n_o$. Now, for $k > n_o$ and $r > r_*$, where $r_* = (R+R_o)/2$ if $R < \infty$ and $r_* = 2R_o$ if $R = \infty$, from (5.2.5) and (5.2.6), we obtain

$$||G-\hat{g}_k||^*_{R_o,\infty} \leq \frac{2M(r,G)}{\Gamma(u+1/2)} \left(\frac{\pi}{2^{2u-1}}\right)^{1/2} \sum_{n=k+1}^{\infty} n^u (R_o/r)^n$$

$$\leq \frac{2M(r,G)}{\Gamma(u+1/2)} \left(\frac{\pi}{2^{2u-1}}\right)^{1/2} (k+1)^u (R_o/r)^{k+1} \sum_{n=0}^{\infty} \left(1+\frac{n}{n_o+1}\right) (R_o/r_*)^n.$$

Since the series $\sum_{n=0}^{\infty} (1+n/(n_o+1))(R_o/r_*)^n$ is convergent, the lemma follows from the above inequality.

LEMMA 5.2.2. <u>Let the GASP</u> $G \varepsilon \bar{G}_{R_o}$, $0 < R_o < \infty$, <u>be given by</u> (5.1.2). <u>Assume that</u> $1 \leq \delta \leq \infty$. <u>Then, for</u> $n \geq 1$, <u>we have</u>

$$(5.2.7) \qquad |b_n| R_o^n \leq K_\infty((n+u)\ \Gamma(n+1)/\Gamma(n+2u))^{1/2}\ \Delta_{n-1,\infty}(G,R_o),$$

$$(5.2.8) \qquad |b_n| R_o^n \leq K_\delta (n+2)(n+u)\ \Delta_{n-1,\delta}(G,R_o),\ 1 \leq \delta < \infty,$$

where b_n's are the coefficients in the expansion (5.1.2) of G, $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, are the approximation errors given by (5.1.5) and (5.1.6), u is the positive number occuring in the equation (5.1.1) and K_δ, $1 \leq \delta \leq \infty$, is a constant independent of n.

PROOF. For the GASP $G \varepsilon \bar{G}_{R_o}$, $0 < R_o < \infty$, given by (5.1.2), from (5.2.2), for any $g \varepsilon \pi^*_{n-1}$ and $r \leq R_o$, we have

$$(5.2.9) \quad b_n r^n \frac{2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)} \left(\frac{\Gamma(u+1/2)}{\Gamma(2u)}\right)^2$$

$$= \int_o^\pi (G(r,\theta)-g(r,\theta))\, C_n^u(\cos\theta) \sin^{2u}\theta \; d\theta \, .$$

Taking, in particular, $r = R_o$ in (5.2.9) and then using Schwartz's inequality and (4.2.7), we obtain

$$|b_n| R_o^n \frac{2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)} \left(\frac{\Gamma(u+1/2)}{\Gamma(2u)}\right)^2 \leq ||G-g||^*_{R_o,\infty} \int_o^\pi |C_n^u(\cos\theta)| \sin^{2u}\theta\, d\theta$$

$$\leq ||G-g||^*_{R_o,\infty} \left(\frac{\pi 2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)}\right)^{1/2} \frac{\Gamma(u+1/2)}{\Gamma(2u)}$$

and this gives

$$(5.2.10) \quad |b_n| R_o^n \leq ||G-g||^*_{R_o,\infty} \left(\frac{\pi(n+u)}{2^{2u-1}} \frac{\Gamma(n+1)}{\Gamma(n+2u)}\right)^{1/2} \frac{\Gamma(2u)}{\Gamma(u+1/2)} \, .$$

By the definition (5.1.5) of the approximation error $\Delta_{n,\infty}(G,R_o)$, for every positive integer n, there exists a GASP polynomial $g_{n-1,\infty} \varepsilon \pi^*_{n-1}$ such that

$$2\, \Delta_{n-1,\infty}(G,R_o) \geq ||G-g_{n-1,\infty}||_{R_o,\infty}$$

and so, from the definition (5.1.3) of uniform norm, we get

$$2\, \Delta_{n-1,\infty}(G,R_o) \geq \frac{1}{J_1} ||G-g_{n-1,\infty}||^*_{R_o,\infty} \, .$$

where $J_1 < \infty$ is such that $1/w_1 \leq J_1$ on $\bar{D}_{R_o}$. From (5.2.10) and the above inequality, for $n \geq 1$, we now get

$$|b_n| R_o^n \leq 2J_1 \frac{\Gamma(2u)}{\Gamma(u+1/2)} \left(\frac{\pi(n+u)}{2^{2u-1}} \frac{\Gamma(n+1)}{\Gamma(n+2u)}\right)^{1/2} \Delta_{n-1,\infty}(G,R_o).$$

This proves (5.2.7) with $K_\infty = 2J_1\ \Gamma(2u)\ \sqrt{\pi}/(\Gamma(u+1/2)\ 2^{u-1/2})$.

Now, from (5.2.9), in view of (5.2.4), for any $r \leq R_o$ and $g \in \pi^*_{n-1}$, we have

$$|b_n| r^n \frac{2^{2u-1}}{(n+u)} \frac{\Gamma(n+2u)}{\Gamma(n+1)} \left(\frac{\Gamma(u+1/2)}{\Gamma(2u)}\right)^2 \leq \frac{\Gamma(n+2u)}{\Gamma(2u)\ \Gamma(n+1)} \int_o^\pi |G(r,\theta)-g(r,\theta)| d\theta$$

Since the GASP's $G(x,y)$ and $g(x,y)$ are even in y, the above relation gives that

$$|b_n| r^n \leq \frac{(n+u)\Gamma(2u)}{2^{2u}\ (\Gamma(u+1/2))^2} \int_o^{2\pi} |G(r,\theta)-g(r,\theta)|\ d\theta .$$

Multiplying both the sides of the above inequality by $r\,dr$ and integrating from 0 to R_o, for any $g \in \pi^*_{n-1}$, we get

$$(5.2.11) \qquad |b_n| \frac{R_o^{n+2}}{n+2} = |b_n| \int_o^{R_o} r^{n+1}\ dr$$

$$\leq \frac{(n+u)\ \Gamma(2u)}{2^{2u}(\Gamma(u+1/2))^2} \iint_{\bar{D}_{R_o}} |G(x,y)-g(x,y)|\,dx\,dy.$$

Now, by the definition (5.1.6) of the approximation error $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta < \infty$, and the definition (5.1.4) of L^δ-norm, for every positive integer n there exists a GASP polynomial $g_{n-1,\delta} \in \pi^*_{n-1}$ such that

$$(5.2.12) \qquad 2\Delta_{n-1,\delta}(G,R_o) \geq ||G-g_{n-1,\delta}||_{R_o,\delta}$$

$$\geq \frac{1}{J_2^{1/\delta}} \left(\iint_{\bar{D}_{R_o}} |G(x,y)-g_{n-1,\delta}(x,y)|^\delta\ dx\ dy\right)^{1/\delta} ,$$

where $J_2 < \infty$ is such that $1/w_2 \le J_2$ on $\bar{D}_{R_o}$. For $\delta = 1$, (5.2.8) follows from (5.2.11) and (5.2.12) with $K_1 = \dfrac{2J_2\ \Gamma(2u)}{R_o^2\ 2^{2u}(\Gamma(u+1/2))^2}$.
Now, if $1 < \delta < \infty$, we choose $\eta > 0$ such that $1/\eta + 1/\delta = 1$. By Holder's inequality, we now have

$$\iint_{\bar{D}_{R_o}} |G(x,y)-g_{n-1,\delta}(x,y)|dx\,dy$$
$$\le \Big(\iint_{\bar{D}_{R_o}} |G(x,y)-g_{n-1,\delta}(x,y)|^{\delta}\ dx\,dy\Big)^{1/\delta}\ \Big(\iint_{\bar{D}_{R_o}} dx\,dy\Big)^{1/\eta}.$$

From (5.2.11), (5.2.12) and the above inequality, for $1 < \delta < \infty$, we get

$$|b_n|\ R_o^n \le \frac{2J_2^{1/\delta}\ \Gamma(2u)(\pi R_o^2)^{1/\eta}}{R_o^2\ 2^{2u}(\Gamma(u+1/2))^2}\ (n+2)(n+u)\ \Delta_{n-1,\delta}(G,R_o).$$

This proves(5.2.8) for the case $1 < \delta < \infty$ also, with $K_\delta = (2J_2^{1/\delta}\ \Gamma(2u)(\pi R_o)^{1/\eta})/(R_o^2\ 2^{2u}(\Gamma(u+1/2))^2)$. The proof of the lemma is thus complete.

We now prove

<u>THEOREM</u> 5.2.1. <u>Let the</u> GASP $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, <u>and let</u> $1 \le \delta \le \infty$. <u>Then</u> R, $R_o < R$, <u>is the largest value of</u> r <u>such that</u> G <u>is regular in</u> D_r, <u>if and only if</u>,

$$(5.2.13)\qquad \limsup_{n\to\infty}\ (\Delta_{n,\delta}(G,R_o))^{1/n} = R_o/R,\ 1 \le \delta \le \infty,$$

<u>where the approximation errors</u> $\Delta_{n,\delta}(G,R_o)$, $1 \le \delta \le \infty$, <u>are given by</u> (5.1.5) <u>and</u> (5.1.6).

<u>PROOF</u>. First, let R be the largest value of r such that G is regular in D_r. We now choose the GASP polynomials $\hat{g}_n \in \pi_n^*$ of

Lemma 5.2.1 satisfying (5.2.1). Using the definitions (5.1.5) and (5.1.6) of the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, we get

(5.2.14) $$\Delta_{n,\delta}(G,R_o) \leq ||G-\hat{g}_n||_{R_o,\delta}, \quad 1 \leq \delta \leq \infty.$$

Thus, by the definitions of the norms $||\cdot||_{R_o,\delta}$, $1 \leq \delta \leq \infty$, given by (5.1.3) and (5.1.4), we have

(5.2.15) $$||G-\hat{g}_n||_{R_o,\delta} \leq K^o_\delta ||G-\hat{g}_n||^*_{R_o,\infty}, \quad 1 \leq \delta \leq \infty,$$

where $K^o_\infty = \max_{(x,y)\in \bar{D}_{R_o}} w_1(x,y)$ and $K^o_\delta = \left(\iint_{\bar{D}_{R_o}} w_2(x,y)\,dx\,dy\right)^{1/\delta}$, $1 \leq \delta < \infty$, are constants. From (5.2.14), (5.2.15) and Lemma 5.2.1, we now have

(5.2.16) $$\Delta_{n,\delta}(G,R_o) \leq K^o_\delta\, K\, M(r,G)(n+1)^u (R_o/r)^{n+1}$$

for all r sufficiently near to R and all sufficiently large values of n. The above relation gives that $\limsup_{n\to\infty}(\Delta_{n,\delta}(G,R_o))^{1/n} \leq R_o/r$, for all r sufficiently near to R, and so

(5.2.17) $$\limsup_{n\to\infty} (\Delta_{n,\delta}(G,R_o))^{1/n} \leq R_o/R.$$

For $R = \infty$, (5.2.17) gives that

(5.2.18) $$\lim_{n\to\infty} (\Delta_{n,\delta}(G,R_o))^{1/n} = 0.$$

For $R_o < R < \infty$, if possible, let

(5.2.19) $$\limsup_{n\to\infty} (\Delta_{n,\delta}(G,R_o))^{1/n} = R_o/R' < R_o/R, \quad R' > R.$$

From (5.1.2), (5.2.4) and Lemma 5.2.2, since $(n+u)\Gamma(n+1)/\Gamma(n+2u) \sim n^{1-2u}$ as $n \to \infty$, for $1 \leq \delta \leq \infty$, we now have

$$(5.2.20) \qquad |\sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)| \leq \frac{1}{\Gamma(2u)} \sum_{n=0}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(n+1)} |b_n| r^n$$

$$\leq |b_o| + \frac{K_\delta K^*}{\Gamma(2u)} \sum_{n=1}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(n+1)}(n+u)(n+2)\, \Delta_{n-1,\delta}(G,R_o)(r/R_o)^1$$

where K^* is a constant. From (5.2.20) it follows that if (5.2.19) holds, then the series (5.1.2) converges uniformly on compact subsets of $D_{R'}$ and so the GASP G is regular in $D_{R'}$. But this is impossible, since D_R is the largest disc centered at the origin in which G is regular and $R' > R$. Thus, the supposition (5.2.19) is false and, in view of (5.2.17), we get

$$\limsup_{n \to \infty} (\Delta_{n,\delta}(G,R_o))^{1/n} = R_o/R, \quad R_o < R < \infty.$$

The necessity part of the theorem follows from (5.2.18) and the above relation.

Conversely, let (5.2.13) hold for $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$. Then, it follows from (5.2.20), that the series expansion (5.1.2) of $G \in \bar{G}_{R_o}$ converges uniformly on compact subsets of D_R and so G is regular in D_R, $R > R_o$. That R is the largest value of r such that G is regular in D_r now follows from the necessity part of the theorem. This proves the sufficiency part of the theorem.

The theorem is thus proved.

We now obtain characterizations of (α,β)-order and lower (α,β)-order of a GASP G, regular in D_R, $0 < R < \infty$, in terms of

the approximation errors $\Delta_{n,\delta}(G,R_o)$, $0 < R_o < R$ and $1 \leq \delta \leq \infty$, given by (5.1.5) and (5.1.6). We first need the following lemma.

LEMMA 5.2.3. *Let a GASP G, regular in D_R, $0 < R < \infty$, be of (α,β)-order $\rho(\alpha,\beta,G)$ and lower (α,β)-order $\lambda(\alpha,\beta,G)$. Assume that $0 < R_o < R$ and $1 \leq \delta \leq \infty$. Then the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, where the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, are defined by (5.1.5) and (5.1.6), is analytic in D_R. For $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$, the (α,β)-order $\rho(\alpha,\beta,g_\delta)$ and lower (α,β)-order $\lambda(\alpha,\beta,g_\delta)$ of $g_\delta(z)$ satisfy*

$$(5.2.21)_a \qquad \rho(\alpha,\beta,G) = P(\rho(\alpha,\beta,g_\delta)),$$

$$(5.2.21)_b \qquad \lambda(\alpha,\beta,G) = P(\lambda(\alpha,\beta,g_\delta)).$$

For the case $\alpha \equiv \beta$ ($\alpha(x) \neq \log x$), $(5.2.21)_a$ continues to hold provided $\rho(\alpha,\alpha,G) \geq 1$ while $(5.2.21)_b$ continues to hold provided $\lambda(\alpha,\alpha,G) \geq 1$.

PROOF. We define a function $g_\delta^o(z)$, $1 \leq \delta \leq \infty$, as

$$(5.2.22) \qquad g_\delta^o(z) = \sum_{n=0}^{\infty} \frac{\Delta_{n,\delta}(G,R_o)}{(n+1)^u} (z/R_o)^{n+1}$$

where u is the positive constant occuring in (5.1.1). By Theorem 5.2.1, the function $g_\delta^o(z)$ is analytic in D_R. For a given b, $0 < b < 1$, from (5.2.16), for all r sufficiently near to R, we have

$$M(r,g_\delta^o) = \sum_{n=0}^{\infty} \frac{\Delta_{n,\delta}(G,R_o)}{(n+1)^u} (r/R_o)^{n+1}$$

$$\leq Q(r) + K_\delta^o K\, M(r+b(R-r),G) \sum_{n=0}^{\infty} \left(\frac{r}{r+b(R-r)}\right)^{n+1}$$

$$\leq Q(r) + \frac{K_\delta^O\, K\, R}{b(R-r)} M(r+b(R-r),G),$$

where $Q(r)$ is a polynomial. The above relation, for all r sufficiently near to R, gives that

$$(5.2.23) \quad \log M(r,g_\delta^O) \leq \log ((K_\delta^O K)/b) + \log \frac{R}{R-r} + \log^+ Q(r) + \log 2 + \log M(r+b(R-r),G)$$

$$\leq \max \{3 \log (R/(R-r)), 3 \log M(r+b(R-r),G)\},$$

since $Q(r)$ is bounded in D_R. Thus, for $\alpha(x) \in \Lambda$ and all r sufficiently near to R, we have

$$(5.2.24) \quad \alpha(\log M(r,g_\delta^O)) \leq \max\{\alpha(3 \log (R/(R-r))), \alpha(3 \log M(r+b(R-r),G))\}.$$

From (5.2.24), for $\alpha(x) \in \Lambda$, $\beta(x) \in L^O$ and all r sufficiently near to R, we obtain

$$(5.2.25) \quad \frac{\alpha(\log M(r,g_\delta^O))}{\beta(R/(R-r))} \leq \max \left\{\frac{\alpha(3 \log (R/(R-r)))}{\beta(R/(R-r))}, \frac{\alpha(3 \log M(r+b(R-r),G))}{\beta(R/(R-r))}\right\}.$$

If $\alpha(x)$ and $\beta(x)$ satisfy (2.1.3) and $\alpha \not\equiv \beta$, then $\alpha(x)/\beta(x) \to 0$ as $x \to \infty$, and so (5.2.25), on proceeding to limits and using Lemma 2.2.1, gives that

$$(5.2.26) \quad \rho(\alpha,\beta,g_\delta^O) \leq P(\rho(\alpha,\beta,G)), \ \lambda(\alpha,\beta,g_\delta^O) \leq P(\lambda(\alpha,\beta,G)).$$

It follows from (5.2.25) that (5.2.26) holds for the cases $\alpha \equiv \beta$ also.

We now consider the function $g_\delta^*(z)$, defined as

$$(5.2.27) \quad g_\delta^*(z) = \sum_{n=1}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(n+1)} (n+2)(n+u)\, \Delta_{n-1,\delta}(G,R_o)(z/R_o)^n.$$

By Theorem 5.2.1, $g_\delta^*(z)$ is analytic in D_R. Further, by (5.2.20), we have

$$(5.2.28) \quad M(r,G) \le |b_o| + \frac{K_\delta K^*}{\Gamma(2u)} \sum_{n=1}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(n+1)} (n+2)(n+u)\Delta_{n-1,\delta}(G,R_o) \times (r/R_o)^n$$

$$= |b_o| + \frac{K_\delta K^*}{\Gamma(2u)} M(r,g_\delta^*).$$

This easily gives that

$$(5.2.29) \quad \rho(\alpha,\beta,G) \le \rho(\alpha,\beta,g_\delta^*),\ \lambda(\alpha,\beta,G) \le \lambda(\alpha,\beta,g_\delta^*).$$

Now, by Theorem 5.2.1, the function $g_\delta(z)$, defined as

$$g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n, \quad 1 \le \delta \le \infty,$$

where the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \le \delta \le \infty$, are given by (5.1.5) and (5.1.6), is analytic in D_R. Since $\Gamma(n+2u)(n+2)(n+u)/\Gamma(n+1) \sim n^{2u+1}$ as $n \to \infty$, on applying $(4.3.21)_a$ and $(4.3.21)_b$ to the functions $g_\delta^o(z)$, $g_\delta^*(z)$ and $g_\delta(z)$, we get

$$(5.2.30)_a \quad \rho(\alpha,\beta,g_\delta^o) \le \rho(\alpha,\beta,g_\delta) \le \rho(\alpha,\beta,g_\delta^*) \le P(\rho(\alpha,\beta,g_\delta^o)),$$

while on applying (4.3.31) to the functions $g_\delta^o(z)$, $g_\delta^*(z)$ and $g_\delta(z)$, we get

$$(5.2.30)_b \quad \lambda(\alpha,\beta,g_\delta^o) \le \lambda(\alpha,\beta,g_\delta) \le \lambda(\alpha,\beta,g_\delta^*) \le P(\lambda(\alpha,\beta,g_\delta^o)).$$

The lemma now follows from (5.2.26), (5.2.29), $(5.2.30)_a$ and $(5.2.30)_b$.

We now have

THEOREM 5.2.2. *Let a* GASP G, *regular in* D_R, $0 < R < \infty$, *be of* (α,β)-*order* $\rho(\alpha,\beta,G)$. *Assume that* $0 < R_o < R$ *and* $1 \leq \delta \leq \infty$. *Then, if* $\alpha(x) = \beta(x) = \log x$ *or* $\alpha \not\equiv \beta$, *we have*

$$(5.2.31) \qquad \rho(\alpha,\beta,G) + X(\alpha,\beta) = \tilde{P}(\theta_{G,\delta})$$

where $X(\alpha,\beta) = 1$ *if* $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ *otherwise*,

$$\theta_{G,\delta} = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(n/\log^+ \Delta_{n,\delta}(G,R_o)(R/R_o)^n)}$$

and the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, *are given by* (5.1.5) *and* (5.1.6). *The equation* (5.2.31) *continues to hold for the case* $\alpha \equiv \beta(\alpha(x) \neq \log x)$ *provided* $\rho(\alpha,\alpha,G) \geq 1$.

PROOF. The theorem follows from Lemma 5.2.3, on applying (1.9.4) and Theorem 2.3.1 to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$.

THEOREM 5.2.3. *Let a* GASP G, *regular in* D_R, $0 < R < \infty$, *be of lower* (α,β)-*order* $\lambda(\alpha,\beta,G)$. *Assume that* $0 < R_o < R$ *and* $1 \leq \delta \leq \infty$. *Then, if* $\alpha \not\equiv \beta$, *we have*

$$(5.2.32) \qquad \lambda(\alpha,\beta,G) + X(\alpha,\beta) = \max_{\{n_k\}} [\tilde{P}(\theta_{G,\delta}(\{n_k\}))],$$

where $X(\alpha,\beta) = 1$ *if* $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ *otherwise*,

$$\theta_{G,\delta}(\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(n_{k-1})}{\beta(n_k/\log^+ \Delta_{n_k,\delta}(G,R_o)(R/R_o)^{n_k})},$$

the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, *are given by* (5.1.5) *and* (5.1.6) *and the maximum in* (5.2.32) *is taken over*

all increasing sequences $\{n_k\}$ of positive integers. For the case $\alpha \equiv \beta(\alpha(x) \neq \log x)$, the relation (5.2.32) continues to hold provided $\lambda(\alpha,\alpha,G) \geq 1$. For the case $\alpha(x) = \beta(x) = \log x$, (5.2.32) holds provided the principal indices $\{n_m\}$ of the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$, analytic in D_R, satisfy the condition that $\log n_{m-1} \sim \log n_m$ as $m \to \infty$.

PROOF. The theorem follows from Lemma 5.2.3 on applying (1.9.6) and Theorem 2.4.5 to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$.

For a GASP G, regular in D_R, $0 < R < \infty$, and having slow rate of growth, the (α,β)-order $\rho(\alpha,\beta,G)$ satisfies $\rho(\alpha,\beta,G) = 0$ if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$ and $\rho(\alpha,\beta,G) \leq 1$ if $\alpha \equiv \beta(\alpha(x) \neq \log x)$ and so the results obtained in Theorems 5.2.2 and 5.2.3 fail to give any specific information about the rate of decay of the degree of approximation of such GASP's. Thus, to study the interrelations between the growth and the degree of approximation of GASP's of zero order, we now obtain characterizations of α-logarithmic order and lower α-logarithmic order of a GASP in terms of the approximation errors given by (5.1.5) and (5.1.6). The following lemma is a key result in the proofs of Theorems 5.2.4 to 5.2.6.

LEMMA 5.2.4. Let a GASP G, regular in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,G)$ and lower α-logarithmic order $\lambda(\alpha,G)$. Assume that $0 < R_o < R$ and $1 \leq \delta \leq \infty$. Further, let the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)\ (z/R_o)^n$, analytic in D_R, be of

α-logarithmic order $\rho(\alpha, g_\delta)$ and lower α-logarithmic order $\lambda(\alpha, g_\delta)$. Then, if $\rho(\alpha, G) \geq 1$, we have

$$\rho(\alpha, G) = \max(1, \rho(\alpha, g_\delta))$$

and if $\lambda(\alpha, G) \geq 1$, we also have

$$\lambda(\alpha, G) = \max(1, \lambda(\alpha, g_\delta)).$$

PROOF. Let the function $g_\delta^o(z)$, $1 \leq \delta \leq \infty$, analytic in D_R, be given by (5.2.22). From (5.2.24), with $b = 1/2$, we have

$$\frac{\alpha(\log M(r, g_\delta^o))}{\alpha(\log(R/(R-r)))} \leq \max \left\{\frac{\alpha(3 \log(R/(R-r)))}{\alpha(\log(R/(R-r)))}, \frac{\alpha(3 \log M(r+(R-r)/2, G))}{\alpha(\log(R/(R-r)))}\right\}$$

for $\alpha(x) \in \Lambda$ and all r sufficiently near to R. Since $\alpha(x) \in \Lambda$, on proceeding to limits, the above relation gives that

(5.2.33) $\quad \rho(\alpha, g_\delta^o) \leq \max(1, \rho(\alpha, G))$, $\lambda(\alpha, g_\delta^o) \leq \max(1, \lambda(\alpha, G))$,

where $\rho(\alpha, g_\delta^o)$ and $\lambda(\alpha, g_\delta^o)$ are, respectively, α-logarithmic order and lower α-logarithmic order of $g_\delta^o(z)$.

Now, let the function $g_\delta^*(z)$, $1 \leq \delta \leq \infty$, analytic in D_R, be given by (5.2.27). Then, from (5.2.28), since $\alpha(x) \in \Lambda$, we have

(5.2.34) $\quad \rho(\alpha, G) \leq \rho(\alpha, g_\delta^*)$, $\lambda(\alpha, G) \leq \lambda(\alpha, g_\delta^*)$.

We now consider the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G, R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$, analytic in D_R. Applying (4.3.43) to the functions $g_\delta^o(z)$, $g_\delta^*(z)$ and $g_\delta(z)$, we get

$$(5.2.35)\quad \begin{aligned} \rho(\alpha,g_\delta^o) &\le \rho(\alpha,g_\delta) \le \rho(\alpha,g_\delta^*) \le \max\,(1,\rho(\alpha,g_\delta^o)) \\ \lambda(\alpha,g_\delta^o) &\le \lambda(\alpha,g_\delta) \le \lambda(\alpha,g_\delta^*) \le \max\,(1,\lambda(\alpha,g_\delta^o)). \end{aligned}$$

The lemma now follows easily from (5.2.33), (5.2.34) and (5.2.35).

We now have

THEOREM 5.2.4. *Let a GASP G, regular in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,G)$ with $\rho(\alpha,G) \ge 1$. Assume that $0 < R_o < R$ and $1 \le \delta \le \infty$. Then*

$$\rho\,(\alpha,G) = \max\,\{1, \limsup_{n\to\infty} \frac{\alpha(\log \Delta_{n,\delta}(G,R_o)(R/R_o)^n)}{\alpha\,(\log n)}\},$$

where the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \le \delta \le \infty$, are given by (5.1.5) and (5.1.6).

PROOF. The theorem follows from Lemma 5.2.4 on applying (3.2.2) to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \le \delta \le \infty$.

THEOREM 5.2.5. *Let a GASP G, regular in D_R, $0 < R < \infty$, be of lower α-logarithmic order $\lambda(\alpha,G)$ with $\lambda(\alpha,G) \ge 1$. Assume that $0 < R_o < R$, $1 \le \delta \le \infty$ and that (3.2.9) and (3.2.10) are satisfied. Further, let $\Delta_{n,\delta}(G,R_o)/\Delta_{n+1,\delta}(G,R_o)$, where the approximation errors $\Delta_{n,\delta}(G,R_o)$ are given by (5.1.5) and (5.1.6), be ultimately a nondecreasing function of n. Then*

$$\lambda(\alpha,G) = \max\,\{1, \liminf_{n\to\infty} \frac{\alpha(\log \Delta_{n,\delta}(G,R_o)(R/R_o)^n)}{\alpha(\log n)}\}.$$

PROOF. In view of Lemma 5.2.4, the theorem follows on applying Theorem 3.2.3 to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \le \delta \le \infty$.

Using Lemma 5.2.4 and applying Theorem 3.2.4 to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$, we obtain the following characterization of lower α-logarithmic order of a GASP G, regular in D_R, $R_o < R < \infty$.

THEOREM 5.2.6. *Let a GASP G, regular in D_R, $0 < R < \infty$, be of lower α-logarithmic order $\lambda(\alpha,G)$ with $\lambda(\alpha,G) \geq 1$. Assume that $0 < R_o < R$, $1 \leq \delta \leq \infty$ and that (3.2.9) and (3.2.10) are satisfied. Further, if the principal indices $\{n_m\}$ of the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, $1 \leq \delta \leq \infty$, where the approximation errors $\Delta_{n,\delta}(G,R_o)$ are given by (5.1.5) and (5.1.6), satisfy the condition that $\alpha(\log n_{m-1}) \sim \alpha(\log n_m)$ as $m \to \infty$, then*

$$\lambda(\alpha,G) = \max\,(1, \theta_{\alpha,\delta}(G))$$

where

$$\theta_{\alpha,\delta}(G) = \max_{\{m_k\}} \{\liminf_{k \to \infty} \frac{\alpha(\log \Delta_{n_k,\delta}(G,R_o)(R/R_o)^{n_k})}{\alpha(\log n_{k+1})}\} \tag{5.2.36}$$

and the maximum in (5.2.36) is taken over all increasing sequences $\{n_k\}$ of positive integers.

REMARK. Let a GASP G be regular in the finite disc D_R, $0 < R < \infty$. Then, for the choices $\alpha(x) = \log_{q-1} x$, $q \geq 2$, and $\beta(x) = \log x$, the (α,β)-order $\rho(\alpha,\beta,G)$ of G, denoted by $\rho_q(G)$, will be called *q-order* of G. Clearly $\rho_2(G) = \rho(G)$. If the q-order $\rho_q(G)$, $q \geq 3$, satisfies $0 < \rho_q(G) < \infty$, then, for the choices $\alpha(x) = \log_{q-2} x$, $\beta(x) = x^{\rho_q(G)}$, $\rho(\alpha,\beta,G)$ and $\lambda(\alpha,\beta,G)$ will be called, respectively, *q-type* and *lower q-type* of G. However, the choice $\alpha(x) = x$ and

$\beta(x) = x^d$, $0 < d < \infty$, is not permissible in Definition 5.1.1. Thu if the order $\rho(G)$ of a GASP G satisfies $0 < \rho(G) < \infty$, we define the <u>type</u> T(G) and <u>lower type</u> t(G) of the GASP G as

$$\begin{matrix} T(G) \\ t(G) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r,G)}{(R/(R-r))^{\rho(G)}} .$$

Using the inequalities (5.2.23) and (5.2.28) and the relations (1.9.7) and (1.9.10), characterizations of type T(G) and lower type t(G) of a GASP G, regular in D_R, $0 < R < \infty$, can be obtained in terms of the approximation errors $\Delta_{n,\delta}(G,R_0)$, $0 < R_0 < R$, $1 \leq \delta \leq \infty$. However, we will not enter into the details.

5.3. The characterizations of the growth parameters, given by Definitions 5.1.1 and 5.1.2, of a GASP G, regular in D_R, $0 < R < \infty$, in terms of the coefficients b_n occuring in the ultra-spherical harmonic expansion (5.1.2) of G have been obtained in this section.

It is known [34 , pp. 165-169] that Gilbert's A_u-operator maps an analytic function $g(\xi)$ of a complex variable ξ onto a GASP G:

$$G(x,y) = A_u(g(\xi)) = K_{(u)} \int_C g(\xi)\ (\zeta-\zeta^{-1})^{2u}\ \zeta^{-1}\ d\zeta \tag{5.3.1}$$

where $\xi = x+iy(\zeta+\zeta^{-1})/2$, $C = \{e^{i\phi} : 0 \leq \phi \leq \pi\}$ and

$$K_{(u)} = \frac{4\Gamma(2u)}{(4i)^{2u}(\Gamma(u))^2} .$$

The function $g(\xi)$ is said to be the A_u-associate of the GASP G. The A_u-associate $g(\xi)$ of a GASP G, with ultra-spherical harmonic expansion (5.1.2), is given by

$$g(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(2u)\ \Gamma(n+1)}\ b_n\ \xi^n . \tag{5.3.2}$$

The inverse operator A_u^{-1} maps a GASP $G(x,y)$ back onto its A_u-associate $g(\xi)$ [34, p. 174] and is given by

(5.3.3) $$g(\xi) = A_u^{-1}(G) = \int_0^{\pi} G(r,\theta)\, K(\xi/r,\theta)\, d\theta\ , \quad |\xi| < r,$$

where

(5.3.4) $$K(\xi/r,\theta) = K^*_{(u)} \frac{(\sin\theta)^{2u}(1-\xi^2/r^2)}{(1-2(\xi/r)\cos\theta + \xi^2/r^2)^{u+1}}$$

and

(5.3.5) $$K^*_{(u)} = \frac{u\ \Gamma(2u)}{2^{2u-1}\ (\Gamma(u+1/2))^2}\ .$$

We need the following lemma.

LEMMA 5.3.1.A ([34, p.180]). *Let a GASP G, regular about the origin, has the ultra-spherical harmonic expansion* (5.1.2). *Then R is the largest value of r such that G is regular in D_r, if and only if,*

$$\limsup_{n\to\infty} |b_n|^{1/n} = 1/R.$$

The following lemma is a key result in the proofs of Theorems 5.3.1 and 5.3.2.

LEMMA 5.3.1. *Let a GASP* $G(r,\theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$, *regular in* D_R, $0 < R < \infty$, *be of* (α,β)-*order* $\rho(\alpha,\beta,G)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,G)$. *Then, the function* $g^*(z) = \sum_{n=0}^{\infty} |b_n|\, z^n$ *is analytic in* D_R. *Further, for* $\alpha(x) = \beta(x) = \log x$ *or* $\alpha \neq \beta$, *the* (α,β)-*order* $\rho(\alpha,\beta,g^*)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,g^*)$ *of* $g^*(z)$ *satisfy,*

$(5.3.6)_a$ $$\rho(\alpha,\beta,G) = P(\rho(\alpha,\beta,g^*)),$$

$(5.3.6)_b \qquad \lambda(\alpha,\beta,G) = P(\lambda(\alpha,\beta,g^*)).$

For the case $\alpha \equiv \beta(\alpha(x) \neq \log x)$, $(5.3.6)_a$ continues to hold provided $\rho(\alpha,\alpha,G) \geq 1$ while $(5.3.6)_b$ continues to hold provided $\lambda(\alpha,\alpha,G) \geq 1$.

PROOF. Since the GASP $G(r,\theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$ is regular in D_R, it follows from Lemma 5.3.1.A that the A_u-associate $g(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(2u)\Gamma(n+1)} b_n \xi^n$ of G is analytic in D_R. For any $r < R$, by the A_u-integral operator (5.3.1), we have

$$(5.3.7) \qquad M(r,G) \leq M(r,g).$$

This gives

$$(5.3.8) \qquad \rho(\alpha,\beta,G) \leq \rho(\alpha,\beta,g),\ \lambda(\alpha,\beta,G) \leq \lambda(\alpha,\beta,g),$$

where $\rho(\alpha,\beta,g)$ and $\lambda(\alpha,\beta,g)$ are respectively, (α,β)-order and lower (α,β)-order of $g(\xi)$.

On the other hand, the inverse integral operator A_u^{-1}, given by (5.3.3), can be written as

$$g(\xi) = A_u^{-1}(G) = \left(\frac{1}{2i}\right)^{2u} K^*_{(u)}(z\bar{z}-\xi^2) \int_{\substack{0\\|z|=\bar{z}}}^{\pi} \frac{V(z,\bar{z})(z-\bar{z})^{2u}}{((z-\xi)(\bar{z}-\xi))^{u+1}}\, d\theta$$

for $|\xi| \leq r < \bar{R} < R$, where $V(z,\bar{z}) = G((z+\bar{z})/2,(z-\bar{z})/(2i))$ and $K^*_{(u)}$ is given by (5.3.5). Taking $\bar{R} = r+b(R-r)$, $0 < b < 1$, the above relation gives that

$$M(r,g) \leq \frac{\left(\frac{1}{2}\right)^{2u} \pi K^*_{(u)}\ 2R^2(2R)^{2u}}{(b(R-r))^{2(u+1)}}\ M(r+b(R-r),G)$$

and so

$$(5.3.9)\quad \log M(r,g) \le \log \frac{2\pi K^*_{(u)}}{b^{2(u+1)}} + 2(u+1)\log\frac{R}{R-r} + \log M(r+b(R-r),$$

$$\le \max\{(2u+3)\log\frac{R}{R-r}, (2u+3)\log M(r+b(R-r),G)\},$$

for all r sufficiently near to R. With (5.3.9) in place of (5.2.23) and proceeding as in the deduction of (5.2.26) from (5.2.23), we obtain

$$(5.3.10)\quad \rho(\alpha,\beta,g) \le P(\rho(\alpha,\beta,G)),\ \lambda(\alpha,\beta,g) \le P(\lambda(\alpha,\beta,G)).$$

We now consider the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$. In view of Lemma 5.3.1.A, $g^*(z)$ is analytic in D_R. Since $\Gamma(n+2u)/\Gamma(n+1) \sim n^{2u-1}$ as $n \to \infty$, on applying $(4.3.23)_a$ and $(4.3.23)_b$ to the functions $g^*(z)$ and $g(\xi)$, we have

$$(5.3.11)\quad \rho(\alpha,\beta,g^*) \le P(\rho(\alpha,\beta,g)) \text{ and } \rho(\alpha,\beta,g) \le P(\rho(\alpha,\beta,g^*)),$$

while on applying (4.3.31) to the functions $g^*(z)$ and $g(\xi)$, we have

$$(5.3.12)\quad \lambda(\alpha,\beta,g^*) \le P(\lambda(\alpha,\beta,g)) \text{ and } \lambda(\alpha,\beta,g) \le P(\lambda(\alpha,\beta,g^*)).$$

The lemma now follows from (5.3.8), (5.3.10), (5.3.11) and (5.3.12).

Using Lemma 5.3.1, we now obtain coefficient characterizations of (α,β)-order and lower (α,β)-order of a GASP.

THEOREM 5.3.1. *Let a* GASP $G(r,\theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$, *regular in* D_R, $0 < R < \infty$, *be of* (α,β)-*order* $\rho(\alpha,\beta,G)$. *Then, if* $\alpha(x) = \beta(x) = \log x$ *or* $\alpha \not\equiv \beta$, *we have*

(5.3.13) $$\rho(\alpha,\beta,G) + X(\alpha,\beta) = \tilde{P}(\theta_G)$$

where $X(\alpha,\beta) = 1$ if $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ otherwise ar

$$\theta_G = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(n/\log^+ |b_n|\ R^n)}.$$

The equation (5.3.13) continues to hold for the case $\alpha \equiv \beta(\alpha(x)$ log x) provided $\rho(\alpha,\alpha,G) \geq 1$.

PROOF. The theorem follows from Lemma 5.3.1 on applying (1.9.4) and Theorem 2.3.1 to the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$.

THEOREM 5.3.2. Let a GASP $G(r,\theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$, regular in D_R, $0 < R < \infty$, be of lower (α,β)-order $\lambda(\alpha,\beta,G)$. Then, if $\alpha \not\equiv \beta$, we have

(5.3.14) $$\lambda(\alpha,\beta,G) + X(\alpha,\beta) = \max_{\{n_k\}} [\tilde{P}(\theta_G(\{n_k\}))]$$

where $X(\alpha,\beta) = 1$ if $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ otherwise,

$$\theta_G(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(n_k/\log^+ |b_{n_k}|\ R^{n_k})}$$

and maximum in (5.3.14) is taken over all increasing sequences $\{n_k\}$ of positive integers. For the case $\alpha \equiv \beta(\alpha(x) \neq \log x)$, the relation (5.3.14) continues to hold provided $\lambda(\alpha,\alpha,G) \geq 1$. For the case $\alpha(x) = \beta(x) = \log x$, (5.3.14) holds provided the principal indices $\{n_m\}$ of the function $g^*(z) = \sum_{n=0}^{\infty} |b_n|\ z^n$ satisfy the condition that $\log n_{m-1} \sim \log n_m$ as $m \to \infty$.

PROOF. The theorem follows from Lemma 5.3.1 on applying (1.9.6) and Theorem 2.4.5 to the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$.

We now obtain coefficient characterizations of α-logarithmi order and lower α-logarithmic order of a GASP. We first need th following key lemma.

LEMMA 5.3.2. *Let a GASP* $G(r,\theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$, *regular in* D_R, $0 < R < \infty$, *be of* α-*logarithmic order* $\rho(\alpha,G)$ *and lower* α-*logarithmic order* $\lambda(\alpha,G)$. *Assume that the function* $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$, *analytic in* D_R, *is of* α-*logarithmic order* $\rho(\alpha,g^*)$ *and lower* α-*logarithmic order* $\lambda(\alpha,g^*)$. *If* $\rho(\alpha,G) \geq 1$, *then we have*

$$\rho(\alpha,G) = \max(1,\rho(\alpha,g^*)),$$

and if $\lambda(\alpha,G) \geq 1$, *we also have*

$$\lambda(\alpha,G) = \max(1,\lambda(\alpha,g^*)).$$

PROOF. Let the function $g(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2u)}{\Gamma(2u)\ \Gamma(n+1)} b_n \xi^n$, analytic in D_R, be the A_u-associate of the given GASP G. Then, by (5.3.7), we have

$$\rho(\alpha,G) \leq \rho(\alpha,g),\ \lambda(\alpha,G) \leq \lambda(\alpha,g) \tag{5.3.15}$$

where $\rho(\alpha,g)$ and $\lambda(\alpha,g)$ are respectively, α-logarithmic order and lower α-logarithmic order of the function $g(\xi)$.

Now, with (5.3.9) in place of (5.2.23) we proceed as in the deduction of (5.2.33) from (5.2.23), to obtain

$$\rho(\alpha,g) \leq \max(1,\rho(\alpha,G)),\ \lambda(\alpha,g) \leq \max(1,\lambda(\alpha,G)); \tag{5.3.16}$$

We now consider the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$, analytic in D_R. Since $\Gamma(n+2u)/\Gamma(n+1) \sim n^{2u-1}$ as $n \to \infty$, it follows from

(4.3.43) that

(5.3.17) $\rho(\alpha, g^*) \le \max(1, \rho(\alpha, g)),\ \rho(\alpha, g) \le \max(1, \rho(\alpha, g^*))$

(5.3.18) $\lambda(\alpha, g^*) \le \max(1, \lambda(\alpha, g)),\ \lambda(\alpha, g) \le \max(1, \lambda(\alpha, g^*))$.

The lemma follows easily from (5.3.15), (5.3.16), (5.3.17) and (5.3.18).

We now have

THEOREM 5.3.3. *Let a GASP* $G(r, \theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$, *regular in* D_R, $0 < R < \infty$, *be of* α-*logarithmic order* $\rho(\alpha, G)$ *with* $\rho(\alpha, G) \ge 1$ *Then*

$$\rho(\alpha, G) = \max\{1, \limsup_{n \to \infty} \frac{\alpha(\log |b_n| R^n)}{\alpha(\log n)}\}.$$

PROOF. The theorem follows easily from Lemma 5.3.2 on applying (3.2.2) to the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$, analytic in D_R.

THEOREM 5.3.4. *Let a GASP* $G(r, \theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta)$, *regular in* D_R, $0 < R < \infty$, *be of lower* α-*logarithmic order* $\lambda(\alpha, G)$ *with* $\lambda(\alpha, G) \ge 1$. *Assume that* (3.2.9) *and* (3.2.10) *are satisfied and that* $|b_n/b_{n+1}|$ *is ultimately a nondecreasing function of* n. *Then*

$$\lambda(\alpha, G) = \max\{1, \liminf_{n \to \infty} \frac{\alpha(\log |b_n| R^n)}{\alpha(\log n)}\}.$$

PROOF. In view of Lemma 5.3.2, the theorem follows on applying Theorem 3.2.3 to the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$, analytic in D_R.

REMARK. Another coefficient characterization of the lower α-logarithmic order of a GASP G, regular in D_R, $0 < R < \infty$, can be obtained on using Lemma 5.3.2 and applying Theorem 3.2.4 to the function $g^*(z) = \sum_{n=0}^{\infty} |b_n| z^n$.

5.4. The generalized biaxisymmetric potentials (GBSP's) are the solutions of the elliptic partial differential equation

$$(5.4.1)\quad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{2u+1}{y}\frac{\partial F}{\partial y} + \frac{2v+1}{x}\frac{\partial F}{\partial x} = 0,\ u,v > -1/2,$$

which are even in x and y. A polynomial of degree n, which is even in x and y, is said to be a GBSP polynomial of degree n, if it satisfies (5.4.1). A GBSP $F \equiv F_{u,v}$, regular about the origin, can be expanded as

$$(5.4.2)\qquad F(x,y) \equiv F(r,\theta) = \sum_{n=0}^{\infty} c_n\, r^{2n}\, P_n^{(u,v)}(\cos 2\theta),$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $P_n^{(u,v)}(t)$ are Jacobi polynomial

A GBSP F is said to be regular in the disc D_R, $0 < R < \infty$, if the series on the right hand side of (5.4.2) converges uniformly on compact subsets of D_R. In particular a GBSP F regular in D_∞ is called an entire GBSP.

To study the growth of GBSP's F, regular in a finite disc D_R, $0 < R < \infty$, we introduce the following growth parameters.

DEFINITION 5.4.1. *A GBSP F, regular in a finite disc* D_R, $0 < R < \infty$, *is said to be of* (α,β)-*order* $\rho(\alpha,\beta,F)$ *and lower* (α,β)-*order* $\lambda(\alpha,\beta,F)$, $0 \le \lambda(\alpha,\beta,F) \le \rho(\alpha,\beta,F) \le \infty$, *if*

$$\begin{matrix} \rho(\alpha,\beta,F) \\ \lambda(\alpha,\beta,F) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,F))}{\beta(R/(R-r))},$$

<u>where</u> $\alpha(x) \in \Lambda$ <u>and</u> $\beta(x) \in L^{O}$ <u>satisfy either of the conditions</u> (2.1.2) <u>and</u> (2.1.3) <u>and</u>

$$M(r,F) = \max_{0 \leq \theta \leq 2\pi} |F(r,\theta)|.$$

<u>REMARK</u>. If (2.1.2) is satisfied, i.e., $\alpha(x) = \beta(x) = \log x$, then $\rho(\alpha,\beta,F)$ and $\lambda(\alpha,\beta,F)$, denoted by $\rho(F)$ and $\lambda(F)$, are called, respectively, <u>order</u> and <u>lower order</u> of the GBSP F, regular in D_R, $0 < R < \infty$. A GBSP F, regular in D_R, $0 < R < \infty$, is said to be of <u>slow growth</u> if $\rho(F) = 0$.

To measure precisely the growth of a GBSP of slow growth we now introduce the concepts of α-logarithmic order and lower α-logarithmic order.

<u>DEFINITION</u> 5.4.2. <u>A</u> GBSP F, <u>regular in</u> D_R, $0 < R < \infty$, <u>is said to be of</u> α-<u>logarithmic order</u> $\rho(\alpha,F)$ <u>and lower</u> α-<u>logarithmic order</u> $\lambda(\alpha,F)$, $0 \leq \lambda(\alpha,F) \leq \rho(\alpha,F) \leq \infty$, <u>if</u>

$$\begin{matrix} \rho(\alpha,F) \\ \lambda(\alpha,F) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,F))}{\alpha(\log (R/(R-r)))}$$

<u>where</u> $\alpha(x) \in \Lambda$ <u>and</u> (3.1.1) <u>is satisfied</u>.

To avoid some trivial cases we shall assume throughout that $M(r,F) \to \infty$ as $r \to R$.

A GBSP F, given by (5.4.2), is said to be regular on the closed disc $\bar{D}_R$, $0 < R < \infty$, if it is regular in some open disc

$D_{R'}$, $R' > R$. Let $\bar{F}_{R_o}$, $0 < R_o < \infty$, be the class of all GBSP's regular on $\bar{D}_{R_o}$, $0 < R_o < \infty$. Let w_1 and w_2 be two positive functions defined on $\bar{D}_{R_o}$, $0 < R_o < \infty$, such that $1/w_i$, $i = 1,2$, are bounded on $\bar{D}_{R_o}$. Further, let w_1 be continuous and w_2 be integrable on $\bar{D}_{R_o}$. For $F \in \bar{F}_{R_o}$, set

$$(5.4.3) \qquad ||F||_{R_o,\infty} = \max_{x^2+y^2 \leq R_o^2} w_1(x,y)\ |F(x,y)|,$$

$$(5.4.4) \qquad ||F||_{R_o,\delta} = \Big(\iint_{\bar{D}_{R_o}} w_2(x,y)\ |F(x,y)|^{\delta}\ dx\ dy\Big)^{1/\delta}, 1 \leq \delta <$$

Then, $||\cdot||_{R_o,\infty}$ and $||\cdot||_{R_o,\delta}$ are, respectively, uniform norm and L^{δ}-norm on $\bar{F}_{R_o}$. In particular, when $w_1 \equiv 1$ on $\bar{D}_{R_o}$ we shall denote $||\cdot||_{R_o,\infty}$ by $||\cdot||^{*}_{R_o,\infty}$.

For $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, we define the approximation errors $\Delta_{n,\infty}(F,R_o)$ and $\Delta_{n,\delta}(F,R_o)$, $1 \leq \delta < \infty$, as

$$(5.4.5) \qquad \Delta_{n,\infty}(F,R_o) = \inf_{f \in \pi_n^o} ||F-f||_{R_o,\infty},$$

$$(5.4.6) \qquad \Delta_{n,\delta}(F,R_o) = \inf_{f \in \pi_n^o} ||F-f||_{R_o,\delta}\ , 1 \leq \delta < \infty,$$

where π_n^o consists of all GBSP polynomials of degree atmost $2n$.

5.5. The influence of the growth of a GBSP F, regular in a finite disc D_R, $0 < R < \infty$, as measured by the growth parameters introduced in Section 5.4, on the rates of decay of the

approximation errors $\Delta_{n,\delta}(F,R_o)$, given by (5.4.5) and (5.4.6), has been investigated in this section.

We first have the following lemmas.

LEMMA 5.5.1. *Let* F *be a* GBSP *regular in* D_R, $0 < R \leq \infty$, *and let* $0 < R_o < R$. *Then, there exist* GBSP *polynomials* $\hat{f}_k \in \pi_k^o$ *such tha*

$$(5.5.1) \qquad ||F-\hat{f}_k||^*_{R_o,\infty} \leq \tilde{K}\, M(r,F)\ (k+1)^{\gamma+1/2}\ (R_o/r)^{2(k+1)}$$

for all r *sufficiently near to* R *and all sufficiently large valu of* k. *Here* $\tilde{K}$ *is a constant independent of* n *and* r *and* $\gamma = \max(u,v)$.

PROOF. Let the GBSP F, regular in D_R, $0 < R \leq \infty$, be given by (5.4.2) and let $\hat{f}_k$ denote the kth partial sum of the expansion (5.4.2) of F, i.e., $\hat{f}_k$ is given by (5.4.2) with with $c_n = 0$ for $n \geq k+1$. Then $\hat{f}_k \in \pi_k^o$ and

$$(5.5.2) \qquad ||F-\hat{f}_k||^*_{R_o,\infty} \leq \sum_{n=k+1}^{\infty} |c_n|\ R_o^{2n}\ |P_n^{(u,v)}(\cos 2\theta)|$$

$$\leq \frac{1}{\Gamma(\gamma+1)} \sum_{n=k+1}^{\infty} |c_n|\ R_o^{2n}\ \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)},$$

since, it is known [122, p. 168] that

$$(5.5.3) \qquad \max_{-1 \leq t \leq 1} |P_n^{(u,v)}(t)| \leq \frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\ \Gamma(n+1)},\ \gamma = \max(u,v).$$

For a GBSP F, given by (5.4.2) and regular in D_R, $0 < R \leq \infty$, we have ([29 , (8)])

$$(5.5.4) \qquad |c_n| \leq \frac{M(r,F)}{r^{2n}}\ [(2n+u+v+1)\ A(n,u,v)\ A(u,v)]^{1/2},$$

for every positive integer n and every $r < R$, where

$$(5.5.5)\quad A(n,u,v) = \frac{\Gamma(n+1)\,\Gamma(n+u+v+1)}{\Gamma(n+u+1)\,\Gamma(n+v+1)}\ ,\ A(u,v) = \frac{\Gamma(u+1)\,\Gamma(v+1)}{\Gamma(u+v+2)}\ .$$

Combining (5.5.2) and (5.5.4), we obtain

$$(5.5.6)\quad ||F-\hat{f}_k||^*_{R_o,\infty} \le \frac{M(r,F)}{\Gamma(\gamma+1)} \sqrt{A(u,v)} \sum_{n=k+1}^{\infty} \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)} \times$$

$$\times\left[(2n+u+v+1)\ A(n,u,v)\right]^{1/2} (R_o/r)^{2n}.$$

Since $\Gamma(x+a)/\Gamma(x) \sim x^a$ as $x \to \infty$, we have

$$\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)} \left[(2n+u+v+1)\ A(n,u,v)\right]^{1/2} \sim \sqrt{2}\ n^{\gamma+1/2}$$

as $n \to \infty$. Thus, for $n > n_o$, we have

$$\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)} \left[(2n+u+v+1)\quad A(n,u,v)\right]^{1/2} \le 2n^{\gamma+1/2}.$$

For $k > n_o$ and $r > r_*$, where $r_* = (R+R_o)/2$ if $R < \infty$ and $r_* = 2R_o$ if $R = \infty$, using (5.5.6) and the above relation, we get

$$||F-\hat{f}_k||^*_{R_o,\infty} \le \frac{M(r,F)}{\Gamma(\gamma+1)}\, 2\sqrt{A(u,v)} \sum_{n=k+1}^{\infty} n^{\gamma+1/2}(R_o/r)^{2n}$$

$$\le \frac{M(r,F)}{\Gamma(\gamma+1)}\quad 2\sqrt{A(u,v)}\,(k+1)^{\gamma+1/2}\ (R_o/r)^{2(k+1)}\times$$

$$\times \sum_{n=0}^{\infty} \left(1 + \frac{n}{n_o+1}\right)^{\gamma+1/2} (R_o/r_*)^{2n}.$$

The lemma follows from the above inequality since the series $\sum_{n=0}^{\infty} \left(1 + \frac{n}{n_o+1}\right)^{\gamma+1/2} (R_o/r_*)^{2n}$ is convergent.

<u>LEMMA</u> 5.5.2. <u>Let the</u> GBSP $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, <u>be given by</u> (5.4.2). <u>Assume that</u> $1 \le \delta \le \infty$. <u>Then, for</u> $n \ge 1$, <u>we have</u>

$$|c_n| R_o^{2n} \leq \tilde{K}_\delta \; \frac{(n+1)(2n+u+v+1)\,\Gamma(n+\gamma+1)A(n,u,v)}{\Gamma(n+1)} \; \Delta_{n-1,\delta}(F,R_o)$$

<u>where</u> $\tilde{K}_\delta$, $1 \leq \delta \leq \infty$, <u>is a constant and</u> $A(n,u,v)$ <u>is given by</u> (5.5.5).

PROOF. It is known ([122, p. 68]) that the Jacobi polynomials $P_n^{(u,v)}$ satisfy the following orthogonality property,

$$(5.5.7) \quad 2\int_0^{\pi/2} \sin^{2u+1}\theta \cos^{2v+1}\theta \, P_n^{(u,v)}(\cos 2\theta)\, P_m^{(u,v)}(\cos 2\theta)\, d\theta = \frac{\delta_m^n}{(2n+u+v+1)A(n,u,v)},$$

where $\delta_m^n = 1$ if $m = n$, $\delta_m^n = 0$ otherwise. From the uniform convergence of the series expansion (5.4.2) of the GBSP F and (5.5.7), we have

$$\frac{c_n r^{2n}}{(2n+u+v+1)A(n,u,v)} = 2\int_0^{\pi/2} (F(r,\theta)-f(r,\theta))P_n^{(u,v)}(\cos 2\theta) \times \times \sin^{2u+1}\theta \cos^{2v+1}\theta \, d\theta$$

for $0 < r \leq R_o$, $n \geq 1$ and any $f \in \pi_{n-1}^o$. Using (5.5.3), the above relation gives that

$$(5.5.8) \quad \frac{|c_n|\, r^{2n}}{(2n+u+v+1)A(n,u,v)} \leq \frac{2\,\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\,\Gamma(n+1)} \int_0^{\pi/2} |F(r,\theta)-f(r,\theta)|\,d\theta .$$

for $0 < r \leq R_o$, $n \geq 1$ and any $f \in \pi_{n-1}^o$. In particular, for $r = R_o$, (5.5.8) gives that

$$(5.5.9) \qquad |c_n| R_o^{2n} \leq \frac{\pi(2n+u+v+1)A(n,u,v)\,\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\,\Gamma(n+1)} \; ||F-f||^*_{R_o,\infty}$$

for $n \geq 1$ and any $f \in \pi_{n-1}^o$. By the definition of the approximation

error $\Delta_{n,\infty}(F,R_o)$, for every positive integer n; there exists a GBSP polynomial $f_{n-1,\infty} \in \pi^o_{n-1}$, such that

$$2\Delta_{n-1,\infty}(F,R_o) \geq ||F-f_{n-1,\infty}||_{R_o,\infty}.$$

Thus, from the definition (5.4.3) of the norm $||\cdot||_{R_o,\infty}$, we have

$$2\Delta_{n-1,\infty}(F,R_o) \geq \frac{1}{J_1} ||F-f_{n-1,\infty}||^*_{R_o,\infty}$$

where $J_1 < \infty$ is such that $1/w_1 \leq J_1$ on $\bar{D}_{R_o}$. For $n \geq 1$, the above inequality, in view of (5.5.9), gives that

$$|c_n| R_o^{2n} \leq \frac{2\pi J_1(2n+u+v+1)A(n,u,v)\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\Gamma(n+1)} \Delta_{n-1,\infty}(F,R_o).$$

For $\delta = \infty$, the lemma follows from the above relation with $\tilde{K}_\infty = 2\pi J_1/\Gamma(\gamma+1)$.

Since the GBSP's F(x,y) and f(x,y) are both even in x and y, from (5.5.8), for $0 < r \leq R_o$ and any $f \in \pi^o_{n-1}$ we have

$$|c_n| r^{2n} \leq \frac{(2n+u+v+1)A(n,u,v)\Gamma(n+\gamma+1)}{2\Gamma(\gamma+1)\Gamma(n+1)} \int_0^{2\pi} |F(r,\theta)-f(r,\theta)|\, d\theta .$$

Multiplying both sides of the above inequality by rdr and integrating from 0 to R_o, for any $f \in \pi^o_{n-1}$, we have

$$(5.5.10)\quad |c_n| R_o^{2n+2} \leq \frac{(n+1)(2n+u+v+1)A(n,u,v)\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\Gamma(n+1)} \times$$

$$\times \iint_{\bar{D}_{R_o}} |F(x,y)-f(x,y)|\,dx\, dy.$$

By the definition (5.4.6) of the approximation error $\Delta_{n,\delta}(F,R_o)$, $1 \leq \delta < \infty$, for every positive integer n, there exists a GBSP

polynomial $f_{n-1,\delta} \varepsilon \pi^{o}_{n-1}$, such that

$$2\Delta_{n-1,\delta}(F,R_o) \geq ||F-f_{n-1,\delta}||_{R_o,\delta}$$

and so, by the definition (5.4.4) of the norm $||\cdot||_{R_o,\delta}$, $1 \leq \delta <$ we have

$$(5.5.11) \quad 2\Delta_{n-1,\delta}(F,R_o) \geq \frac{1}{J_2^{1/\delta}} (\iint_{\bar{D}_{R_o}} |F(x,y)-f_{n-1,\delta}(x,y)|^{\delta} \, dx \, dy$$

where $J_2 < \infty$ is such that $1/w_2 \leq J_2$ on $\bar{D}_{R_o}$. For $\delta = 1$, the lemma follows from (5.5.10) and (5.5.11) with $\tilde{K}_1 = 2J_2/(\Gamma(\gamma+1)R_o^2$ Now, if $1 < \delta < \infty$, we choose $\eta > 0$ such that $1/\eta + 1/\delta = 1$. Using Holder's inequality, we now have

$$(5.5.12) \quad \iint_{\bar{D}_{R_o}} |F(x,y)-f_{n-1,\delta}(x,y)| dx \, dy$$

$$\leq (\iint_{\bar{D}_{R_o}} |F(x,y)-f_{n-1,\delta}(x,y)|^{\delta} dx \, dy)^{1/\delta} (\iint_{\bar{D}_{R_o}} dx \, dy)^{1/\eta}$$

From (5.5.10), (5.5.11) and (5.5.12), for $1 < \delta < \infty$, we get

$$|c_n| R_o^{2n} \leq \frac{2J_2^{1/\delta}}{R_o^2} \frac{(n+1)(2n+u+v+1)A(n,u,v)\,\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\,\Gamma(n+1)} \times$$

$$\times (\pi R_o^2)^{1/\eta} \, \Delta_{n-1,\delta}(F,R_o).$$

This proves the lemma for $1 < \delta < \infty$ also, with $\tilde{K}_\delta = \dfrac{2J_2^{1/\delta}(\pi R_o^2)^{1/\eta}}{R_o^2 \, \Gamma(\gamma+1)}$. The lemma is thus proved.

We now have

THEOREM 5.5.1. *Let the GBSP* $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$ *and let* $1 \le \delta \le \infty$. *Then,* R, $R_o < R$, *is the largest value of* r *such that* F *is regular in* D_r, *if and only if,*

$$(5.5.13) \qquad \limsup_{n \to \infty} (\Delta_{n,\delta}(F,R_o))^{1/2n} = R_o/R, \ 1 \le \delta \le \infty,$$

where the approximation errors $\Delta_{n,\delta}(F,R_o)$, $1 \le \delta \le \infty$, *are given by* (5.4.5) *and* (5.4.6).

PROOF. First, let R be the largest value of r such that the GBSP F is regular in D_r. We now choose the GBSP polynomials $\hat{f}_n \in \pi_n^o$ of Lemma 5.5.1 satisfying (5.5.1). From the definitions (5.4.5) and (5.4.6) of the approximation errors $\Delta_{n,\delta}(F,R_o)$, $1 \le \delta \le \infty$, we obtain

$$\Delta_{n,\delta}(F,R_o) \le ||F-\hat{f}_n||_{R_o,\delta}, \ 1 \le \delta \le \infty,$$

and so, by the definitions (5.4.3) and (5.4.4) of the norms $||\cdot||_{R_o,\delta}$, $1 \le \delta \le \infty$, we have

$$(5.5.14) \qquad \Delta_{n,\delta}(F,R_o) \le \tilde{K}_\delta^o \ ||F-\hat{f}_n||^*_{R_o,\infty}$$

where $\tilde{K}_\infty^o = \max_{(x,y) \in \bar{D}_{R_o}} w_1(x,y)$ and $\tilde{K}_\delta^o = (\iint_{\bar{D}_{R_o}} w_2(x,y)\,dx\,dy)^{1/\delta}$, $1 \le \delta < \infty$, are constants. From (5.5.14) and Lemma 5.5.1, for all r sufficiently near to R and all sufficiently large values of n, we have

$$(5.5.15) \qquad \Delta_{n,\delta}(F,R_o) \le \tilde{K}\,\tilde{K}_\delta^o\, M(r,F)(n+1)^{\gamma+1/2}(R_o/r)^{2(n+1)}.$$

This easily gives that $\limsup_{n\to\infty}(\Delta_{n,\delta}(F,R_o))^{1/2n} \le R_o/r$, for all r sufficiently near to R, and so

(5.5.16) $$\limsup_{n\to\infty} (\Delta_{n,\delta}(F,R_o))^{1/2n} \le R_o/R.$$

For $R = \infty$, the necessity part of the theorem follows from the above inequality. For $R_o < R < \infty$, if possible, let

(5.5.17) $$\limsup_{n\to\infty} (\Delta_{n,\delta}(F,R_o))^{1/2n} = R_o/R' < R_o/R,\ R' > R.$$

From (5.4.2), (5.5.3) and Lemma 5.5.2, for $1 \le \delta \le \infty$, we now get

(5.5.18) $$\left|\sum_{n=0}^{\infty} c_n r^{2n} P_n^{(u,v)}(\cos 2\theta)\right| \le \frac{1}{\Gamma(\gamma+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)} |c_n| r^{2n}$$

$$\le |c_o| + \frac{\tilde{K}_\delta}{\Gamma(\gamma+1)} \sum_{n=1}^{\infty} (n+1)(2n+u+v+1)\left(\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\right)^2 \times$$

$$\times A(n,u,v)\ \Delta_{n-1,\delta}(F,R_o)(r/R_o)^{2n},$$

where $\tilde{K}_\delta$, $1 \le \delta \le \infty$, is a constant. If (5.5.17) holds, then it follows, from (5.5.18), that the series (5.4.2) of F converges uniformly on compact subsets of $D_{R'}$, and so F is regular in $D_{R'}$. But this is impossible, since D_R is the largest disc centered at the origin in which F is regular and $R' > R$. Thus (5.5.17) can not hold and, from (5.5.16), we get

$$\limsup_{n\to\infty} (\Delta_{n,\delta}(F,R_o))^{1/2n} = R_o/R,\quad R_o < R < \infty.$$

For $R_o < R < \infty$, the necessity part of the theorem follows from the above relation.

Conversely, let (5.5.13) hold for a GBSP $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$. Then, from (5.5.18), it follows that the series (5.4.2) of F converges uniformly on compact subsets of D_R and so F is regular in D_R. From the necessity part of the theorem,

it now follows that R is the largest value of r such that F is regular in D_r. The sufficiency part of the theorem is thus prov

This proves the theorem.

We now obtain the interrelations between the (α,β)-order and lower (α,β)-order of a GBSP F, regular in D_R, $0 < R < \infty$, and the approximation errors given by (5.4.5) and (5.4.6). We first prove the following key lemma.

LEMMA 5.5.3. *Let a GBSP F, regular in D_R, $0 < R < \infty$, be of (α,β)-order $\rho(\alpha,\beta,F)$ and lower (α,β)-order $\lambda(\alpha,\beta,F)$. Assume that $0 < R_o < R$ and $1 \leq \delta \leq \infty$. Then the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}$, where $\Delta_{n,\delta}(F,R_o)$, $1 \leq \delta \leq \infty$, are the approximation errors given by (5.4.5) and (5.4.6), is analytic in D_R. For $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$, the (α,β)-order $\rho(\alpha,\beta,f_\delta)$ and lower (α,β)-order $\lambda(\alpha,\beta,f_\delta)$ of $f_\delta(z)$ satisfy*

$$(5.5.19)_a \qquad \rho(\alpha,\beta,F) = P(\rho(\alpha,\beta,f_\delta)),$$

$$(5.5.19)_b \qquad \lambda(\alpha,\beta,F) = P(\lambda(\alpha,\beta,f_\delta)).$$

For the case $\alpha \equiv \beta(\alpha(x) \neq \log x)$, $(5.5.19)_a$ continues to hold if $\rho(\alpha,\alpha,F) \geq 1$ while $(5.5.19)_b$ continues to hold if $\lambda(\alpha,\alpha,F) \geq 1$.

PROOF. Let the function $f_\delta^o(z)$, $1 \leq \delta \leq \infty$, be defined as

$$(5.5.20) \qquad f_\delta^o(z) = \sum_{n=0}^{\infty} \frac{\Delta_{n,\delta}(F,R_o)}{(n+1)^{\gamma+1/2}} (z/R_o)^{2(n+1)}$$

where $\gamma = \max(u,v)$, u and v being the constants occuring in the equation (5.4.1). By Theorem 5.5.1, $f_\delta^o(z)$ is analytic in D_R.

For a given b, $0 < b < 1$, using (5.5.15), for all r sufficiently near to R, we get

$$M(r,f_\delta^O) = \sum_{n=O}^{\infty} \frac{\Delta_{n,\delta}(F,R_O)}{(n+1)^{\gamma+1/2}} (r/R_O)^{2(n+1)}$$

$$= \tilde{Q}(r) + \tilde{K}\,\tilde{K}_\delta^O\, M(r+b(R-r),F) \sum_{n=O}^{\infty} \left(\frac{r}{r+b(R-r)}\right)^{2(n+1)}$$

$$\leq \tilde{Q}(r) + \tilde{K}\,\tilde{K}_\delta^O \frac{R}{b(R-r)} \; M(r+b(R-r),F)$$

where $\tilde{Q}(r)$ is a polynomial. Since $\tilde{Q}(r)$ is bounded in D_R, the above relation, for all r sufficiently near to R, gives that

(5.5.21) $$\log M(r,f_\delta^O) \leq \log(\tilde{K}\,\tilde{K}_\delta^O/b) + \log(R/(R-r)) + \log^+ \tilde{Q}(r)$$
$$+ \log 2 + \log M(r+b(R-r),F)$$
$$\leq \max\left\{3 \log \frac{R}{R-r}\ ,\ 3 \log M(r+b(R-r),F)\right\}.$$

With (5.5.21) in place of (5.2.23) we proceed as in the deduction of (5.2.26) from (5.2.23), to obtain

(5.5.22) $$\rho(\alpha,\beta,f_\delta^O) \leq P(\rho(\alpha,\beta,F)),\quad \lambda(\alpha,\beta,f_\delta^O) \leq P(\lambda(\alpha,\beta,F)).$$

We now define a function $f_\delta^*(z)$, $1 \leq \delta \leq \infty$, as

(5.5.23) $$f_\delta^*(z) = \sum_{n=1}^{\infty} (n+1)(2n+u+v+1)\left(\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\right)^2 \times$$
$$\times A(n,u,v)\ \Delta_{n-1,\delta}(F,R_O)(z/R_O)^{2n}.$$

The function $f_\delta^*(z)$, by Theorem 5.5.1, is analytic in D_R. Further, from (5.5.18), we have

(5.5.24) $$M(r,F) \leq |c_O| + \frac{\tilde{K}_\delta}{\Gamma(\gamma+1)} M(r,f_\delta^*).$$

The above relation gives that

$$(5.5.25)\qquad \rho(\alpha,\beta,F) \le \rho(\alpha,\beta;f_\delta^*),\ \lambda(\alpha,\beta;F) \le \lambda(\alpha;\beta,f_\delta^*).$$

Now, consider the function $f_\delta(z)$, $1 \le \delta \le \infty$; given as

$$f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)\ (z/R_o)^{2n}.$$

Then $f_\delta(z)$ is analytic in D_R, by Theorem 5.5.1. Since, as $n \to \infty$, we have $(n+1)(2n+u+v+1)\ (\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)})^2\ A(n,u,v) \sim 2n^{2+2\gamma}$, on applying $(4.3.23)_a$, $(4.3.23)_b$ and (4.3.31) to the functions $f_\delta^o(z)$, $f_\delta^*(z)$ and $f_\delta(z)$, it follows that

$$(5.5.26)_a\qquad \rho(\alpha,\beta,f_\delta^o) \le \rho(\alpha,\beta,f_\delta) \le \rho(\alpha,\beta,f_\delta^*) \le P(\rho(\alpha,\beta,f_\delta^o)),$$

$$(5.5.26)_b\qquad \lambda(\alpha,\beta,f_\delta^o) \le \lambda(\alpha,\beta,f_\delta) \le \lambda(\alpha,\beta,f_\delta^*) \le P(\lambda(\alpha,\beta,f_\delta^o)).$$

The lemma now follows easily from (5.5.22), (5.5.25), $(5.5.26)_a$ and $(5.5.26)_b$.

We now have

THEOREM 5.5.2. *Let a GBSP F, regular in D_R, $0 < R < \infty$, be of (α,β)-order $\rho(\alpha,\beta,F)$. Assume that $0 < R_o < R$ and $1 \le \delta \le \infty$. Then, if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$, we have*

$$(5.5.27)\qquad \rho(\alpha,\beta,F) + X(\alpha,\beta) = \bar{P}(\theta_{F,\delta}),$$

where $X(\alpha,\beta) = 1$ if $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ *otherwise,*

$$\theta_{F,\delta} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(2n/\log^+ \Delta_{n,\delta}(F,R_o)(R/R_o)^{2n})}$$

and the approximation errors $\Delta_{n,\delta}(F,R_o)$, $1 \le \delta \le \infty$, *are given b*
(5.4.5) *and* (5.4.6). *The equation* (5.5.27) *continues to hold for the case* $\alpha \equiv \beta(\alpha(x) \ne \log x)$ *provided* $\rho(\alpha,\alpha,F) \ge 1$.

PROOF. The theorem follows from Lemma 5.5.3, on applying (1.9.4 and Theorem 2.3.1 to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}$, $1 \le \delta \le \infty$.

THEOREM 5.5.3. *Let a* GBSP F, *regular in* D_R, $0 < R < \infty$, *be of lower* (α,β)-*order* $\lambda(\alpha,\beta,F)$. *Assume that* $0 < R_o < R$ *and* $1 \le \delta \le \infty$. *Then, if* $\alpha \not\equiv \beta$, *we have*

$$\lambda(\alpha,\beta,F) + X(\alpha,\beta) = \max_{\{n_k\}} \left[\bar{P}(\theta_{F,\delta}(\{n_k\})) \right], \tag{5.5.28}$$

where $X(\alpha,\beta) = 1$ if $\alpha(x) = \beta(x) = \log x$, $X(\alpha,\beta) = 0$ *otherwise*,

$$\theta_{F,\delta}(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(2n_{k-1})}{\beta(2n_k/\log^+ \Delta_{n_k,\delta}(F,R_o)(R/R_o)^{2n_k})},$$

the approximation errors $\Delta_{n,\delta}(F,R_o)$, $1 \le \delta \le \infty$, *are given by* (5.4.5) *and* (5.4.6), *and the maximum in* (5.5.28) *is taken over all increasing sequences* $\{n_k\}$ *of positive integers. For the case* $\alpha \equiv \beta(\alpha(x) \ne \log x)$, *the relation* (5.5.28) *continues to hold provided* $\lambda(\alpha,\alpha,F) \ge 1$. *For the case* $\alpha(x) = \beta(x) = \log x$, (5.5.28) *holds provided the principal indices* $\{2n_m\}$ *of the function* $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}$, *analytic in* D_R, *satisfy the condition that* $\log n_{m-1} \sim \log n_m$ *as* $m \to \infty$.

PROOF. The theorem follows from Lemma 5.5.3, on applying (1.9.6) and Theorem 2.4.5 to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}$, $1 \le \delta \le \infty$.

For a GBSP F, regular in D_R, $0 < R < \infty$, and having slow rate of growth, the (α,β)-order $\rho(\alpha,\beta,F)$ satisfies $\rho(\alpha,\beta,F) = 0$ if $\alpha(x) = \beta(x) = \log x$ or $\alpha \not\equiv \beta$ and $\rho(\alpha,\beta,F) \leq 1$ if $\alpha \equiv \beta(\alpha(x) \not\equiv \log x)$ and so the results obtained in Theorems 5.5.2 and 5.5.3 fail to give any specific information about the rate of decay of the degree of approximation of such GBSP's. Thus, to study the interrelations between the growth and the degree of approximation of GBSP's of zero order, we now obtain characterizations of α-logarithmic order and lower α-logarithmic order of a GBSP in terms of the approximation errors given by (5.4.5) and (5.4.6). We first prove the following lemma, which is the main result used in the proofs of Theorems 5.5.4 and 5.5.5.

LEMMA 5.5.4. *Let a GBSP F, regular in D_R, $0 < R < \infty$, be of α-logarithmic order $\rho(\alpha,F)$ and lower α-logarithmic order $\lambda(\alpha,F)$. Assume that $0 < R_0 < R$ and $1 \leq \delta \leq \infty$. Let the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_0)(z/R_0)^{2n}$, analytic in D_R, be of α-logarithmic order $\rho(\alpha,f_\delta)$ and lower α-logarithmic order $\lambda(\alpha,f_\delta)$. Then, if $\rho(\alpha,F) \geq 1$, we have*

$$\rho(\alpha,F) = \max(1,\rho(\alpha,f_\delta)),$$

and if $\lambda(\alpha,F) \geq 1$, we also have

$$\lambda(\alpha,F) = \max(1,\lambda(\alpha,f_\delta)).$$

PROOF. Let the function $f_\delta^0(z)$, analytic in D_R, be defined by (5.5.20). With (5.5.21) in place of (5.2.23) we proceed as in the deduction of (5.2.33) from (5.2.23) to obtain

(5.5.29) $\rho(\alpha, f_\delta^0) \leq \max(1, \rho(\alpha, F)), \ \lambda(\alpha, f_\delta^0) \leq \max(1, \lambda(\alpha, F))$.

Now, let the function $f_\delta^*(z)$, analytic in D_R, be defined by (5.5.23). Then, from (5.5.24), we get

(5.5.30) $\rho(\alpha, F) \leq \rho(\alpha, f_\delta^*), \ \lambda(\alpha, F) \leq \lambda(\alpha, f_\delta^*)$.

We now consider the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F, R_o)(z/R_o)$ analytic in D_R. Applying (4.3.43) to the functions $f_\delta^0(z), f_\delta^*(z)$ and $f_\delta(z)$, we get

$$(5.5.31) \quad \begin{aligned} \rho(\alpha, f_\delta^0) &\leq \rho(\alpha, f_\delta) \leq \rho(\alpha, f_\delta^*) \leq \max(1, \rho(\alpha, f_\delta^0)) \\ \lambda(\alpha, f_\delta^0) &\leq \lambda(\alpha, f_\delta) \leq \lambda(\alpha, f_\delta^*) \leq \max(1, \lambda(\alpha, f_\delta^0)). \end{aligned}$$

The lemma follows from (5.5.29), (5.5.30) and (5.5.31).

We now have

<u>THEOREM</u> 5.5.4. <u>Let a</u> GBSP F, <u>regular in</u> D_R, $0 < R < \infty$, <u>be of</u> α-<u>logarithmic order</u> $\rho(\alpha, F)$ <u>with</u> $\rho(\alpha, F) \geq 1$. <u>Assume that</u> $0 < R_o < R$ <u>and</u> $1 \leq \delta \leq \infty$. <u>Then</u>

$$\rho(\alpha, F) = \max\{1, \limsup_{n \to \infty} \frac{\alpha(\log \Delta_{n,\delta}(F, R_o)(R/R_o)^{2n})}{\alpha(\log n)}\},$$

<u>where the approximation errors</u> $\Delta_{n,\delta}(F, R_o)$, $1 \leq \delta \leq \infty$, <u>are given by</u> (5.4.5) <u>and</u> (5.4.6).

<u>PROOF</u>. The theorem follows from Lemma 5.5.4, on applying (3.2.2) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F, R_o)(z/R_o)^{2n}$, $1 \leq \delta \leq \infty$.

<u>THEOREM</u> 5.5.5. <u>Let a</u> GBSP F, <u>regular in</u> D_R, $0 < R < \infty$, <u>be of lower</u> α-<u>logarithmic order</u> $\lambda(\alpha, F)$ <u>with</u> $\lambda(\alpha, F) \geq 1$. <u>Assume that</u>

$0 < R_0 < R$, $1 \le \delta \le \infty$ and that (3.2.9) and (3.2.10) are satisfi[ed.] Further, if the principal indices $\{2n_m\}$ of the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_0)(z/R_0)^{2n}$, where the approximation errors $\Delta_{n,\delta}(F,R_0)$ are given by (5.4.5) and (5.4.6), satisfy the condition that $\alpha(\log n_{m-1}) \sim \alpha(\log n_m)$ as $m \to \infty$, then

$$\lambda(\alpha,F) = \max\,(1,\theta_{\alpha,\delta}(F))$$

where

$$(5.5.32) \qquad \theta_{\alpha,\delta}(F) = \max_{\{n_k\}} \{\liminf_{k \to \infty} \frac{\alpha(\log \Delta_{n_k,\delta}(F,R_0)(R/R_0)^{2n_k})}{\alpha(\log n_{k+1})}\}$$

and the maximum in (5.5.32) is taken over all increasing sequences $\{n_k\}$ of positive integers.

PROOF. The theorem follows from Lemma 5.5.4, on applying Theorem 3.2.4 to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_0)(z/R_0)^{2n}$, $1 \le \delta \le \infty$.

REMARK. Let a GBSP F be regular in D_R, $0 < R < \infty$. Then, for the choices $\alpha(x) = \log_{q-1} x$, $q \ge 2$, and $\beta(x) = \log x$, the (α,β)-order $\rho(\alpha,\beta,F)$ of F, denoted by $\rho_q(F)$, will be called q-order of F. Clearly $\rho_2(F) = \rho(F)$. If the q-order $\rho_q(F)$, $q \ge 3$, satisfies $0 < \rho_q(F) < \infty$, then, for the choices $\alpha(x) = \log_{q-2} x$ and $\beta(x) = x^{\rho_q(F)}$, $\rho(\alpha,\beta,F)$ and $\lambda(\alpha,\beta,F)$ will be called, respectively, q-type and lower q-type of F. However, the choice $\alpha(x) = x$ and $\beta(x) = x^d$, $0 < d < \infty$, is not permissible in Definition 5.4.1. Thus, if the order $\rho(F)$ of a GBSP F satisfies

$0 < \rho(F) < \infty$, we define the <u>type</u> $T(F)$ and <u>lower type</u> $t(F)$ of F as

$$\begin{matrix} T(F) \\ t(F) \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r,F)}{(R/(R-r))^{\rho(F)}} \, .$$

Using the inequalities (5.5.21) and (5.5.24) and the relations (1.9.8) and (1.9.10), characterizations of type $T(F)$ and lower type $t(F)$ of a GBSP F, regular in D_R, $0 < R < \infty$, can be obtained in terms of the approximation errors $\Delta_{n,\delta}(F,R_o)$, $0 < R_o < R$ and $1 \leq \delta \leq \infty$. However, we will not enter into the details.

CHAPTER 6

GENERALIZED ORDERS OF AN ENTIRE FUNCTION OF SLOW GROWTH

6.1. Let f(z) be an entire function. For the study of the growth of f(z) in a general setting, analogous to that done in Chapters 2 and 3 of the present work, Šeremeta [93] and Shah [102] introduced and investigated the concepts of generalized (α,β)-order and generalized lower (α,β)-order of f(z), given in (1.3.3). As observed in Section 1.3, the choice $\alpha(x) = \log x$ and $\beta(x) = x$ in (1.3.3) gives the classical order $\rho_\infty(f)$ and lower order $\lambda_\infty(f)$ of the entire function f(z), given by (1.2.2) and (1.2.4). An entire function f(z) for which $\rho_\infty(f) = 0$ is said to be of slow growth. Various authors (e.g. [40], [45], [103]) have studied entire functions of slow growth by comparing the growth of $\log_p M(r,f)$, $p \geq 2$, with the growth of $\log_p r$ as $r \to \infty$.

The characterizations of the generalized (α,β)-order and generalized lower (α,β)-order of an entire function, given by (1.3.7), (1.3.10), (1.3.11) and (1.3.12) and obtained by Šeremeta [93] and Shah [102], are derived under the condition

$$\frac{d\,\beta^{-1}(\alpha(x))}{d\log x} = O(1) \text{ as } x \to \infty.$$

Clearly, the above condition is not satisfied for $\beta(x) = \alpha(x)$. Thus, in this case the corresponding results of Šeremeta and Shah are not applicable. In particular, the results obtained in

[40 ,45 ,103] for entire functions of slow growth are not covered by the results of Šeremeta [93] and Shah [102] .

In the present chapter, we define the generalized orders of an entire function in a new way that is specially suited to entire functions of slow growth. Our results, found in this chapter generalize several earlier results ([45,103] etc.), supplement the results of Shah [102] and find applications in Chapter 7.

Let Ω be the class of functions $h(x)$ satisfying the following conditions (A,i) and (A,ii) :

(A,i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$.

(A,ii) There exists a $\theta(x) \varepsilon \Lambda_*$(For the definition of the class Λ_* see Section 1.3), and constants K_1 and K_2, $0 < K_1 \leq K_2 < \infty$ such that

$$K_1 \leq \frac{d\ h(x)}{d\theta(\log x)} \leq K_2$$

for all sufficiently large values of x.

Let $\bar{\Omega}$ be the class of functions satisfying (A,i) and (A,iii) :

$$\text{(A,iii)} \quad \lim_{x \to \infty} \frac{dh(x)}{d \log x} = K,\ 0 < K < \infty.$$

It is easily seen that Ω and $\bar{\Omega}$ are contained in Λ_*. Further, Ω and $\bar{\Omega}$ have no common element.

Let

$$F_{q,j}(x) = \exp_q\ (c \log_{j+1} x),$$

where $\exp_0(w) = w$ and $\exp_q(w) = \exp(\exp_{q-1} w)$, $q = 1,2,\ldots$. Then, the functions $F_{o,p}$, $F_{2,p+1}$, with $0 < c < 1$ if $p = 1$ and $0 < c < \infty$ if $p > 1$, and $F_{p,p}$, $p \geq 2$, $0 < c < 1$, are in Ω with the choices of $\theta(x)$ as $F_{o,p-1}, F_{2,p}$ and $F_{p,p-1}$ respectively. Infact, all the functions of the form $\theta(\log x)$ where $\theta(x) \in \Lambda_*$ are in Ω . The functions $(c_1-c_2/x) \log x$, $c_1 > 0$, $c_2 \geq 0$ and $\log x + c_3(\log_p x)^{c_4}$ where $0 < c_3 < \infty$ and $c_4 \leq 1$ if $p = 1$ and $0 < c_4 < \infty$ if $p > 1$, are in $\bar{\Omega}$. Since $h(x) \in \Omega$, using generalized mean value theorem, we get $h(x^2) \sim h(x)$ as $x \to \infty$. Thus, the functions $F_{1,1}$ with $0 < c <$ and $h_1(x) \log x$, where $h_1(x)$ satisfies (A,i), are neither in Ω nor in $\bar{\Omega}$.

We now have the following definition:

DEFINITION 6.1.1. *An entire function $f(z)$ is said to be of generalized (α,α)-order $\rho_\infty(\alpha,\alpha,f)$ and generalized lower (α,α)-order $\lambda_\infty(\alpha,\alpha,f)$ if*

$$(6.1.1)\qquad \begin{matrix}\rho_\infty(\alpha,\alpha,f)\\ \lambda_\infty(\alpha,\alpha,f)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\alpha(\log r)}$$

where $\alpha(x)$ either belongs to Ω or to $\bar{\Omega}$ and

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

We note here that if $\alpha(x) \in \bar{\Omega}$ then the generalized orders of $f(z)$, given by (6.1.1) coincide with its (2,2)-orders (see [45]). There are certain functions, e.g., $(\log x)^c$, $0 < c < 1$, or $(\log x)^{c_1} (\log_2 x)^{c_2} \ldots (\log_p x)^{c_p}$ where $c_1 \geq 1$ and atleast one c_i, $i = 2,\ldots,p$ nonzero if $c_1 = 1$, which are inadmissible

in Ω or $\bar{\Omega}$ but if the rate of growth of an entire function f(z) with respect to such functions is measured by (6.1.1) with $1 < \rho_{\infty}(\alpha,\alpha,f) < \infty$, then the same is as well measured by the functions in $\bar{\Omega}$. Thus, excluding these functions from the classes Ω or $\bar{\Omega}$ does not mean excluding entire functions of certain types of growth from our discussion.

We also have the following definition:

DEFINITION 6.1.2. *An entire function f(z) with generalized (α,α)-order $\rho_{\infty}(\alpha,\alpha,f)$ and generalized lower (α,α)-order $\lambda_{\infty}(\alpha,\alpha,f)$ is said to be of generalized regular (α,α)-growth if $\rho_{\infty}(\alpha,\alpha,f) = \lambda_{\infty}(\alpha,\alpha,f)$ and f(z) is said to be of generalized irregular (α,α)-growth if it is not of generalized regular (α,α)-growth.*

We shall use the following notations throughout the rest of work.

NOTATION 6.1.1.

$$Q_{\tau}(\xi) = \begin{cases} \max(1,\xi) \text{ if } \alpha(x) \in \Omega \\ \tau+\xi \text{ if } \alpha(x) \in \bar{\Omega} \end{cases}$$

where $\xi \equiv \xi(\alpha)$ and $\tau \equiv \tau(\alpha)$ are functions of $\alpha(x)$. We shall write $Q(\xi)$ for $Q_1(\xi)$.

NOTATION 6.1.2. $\bar{F}(x,c) = \alpha^{-1}(c\alpha(x)),\ 0 < c < \infty$.

In Section 6.2, we obtain the characterizations of generalized (α,α)-order of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, $a_n \neq 0$ for all n, in terms of the coefficients a_n. We also obtain characterizations of generalized (α,α)-order and generalized lower

(α,α)-order of an entire function in terms of its maximum term and the central index, in this section. Section 6.3 deals with the coefficient characterizations of generalized lower (α,α)-orde In the same section, we also find a necessary and sufficient condition for an entire function $f(z)$ to be of generalized regula (α,α)-growth. In Section 6.4, a decomposition theorem for an entire function of generalized irregular (α,α)-growth is found.

Throughout the rest of the work we shall assume that $\alpha(x) \in \Lambda_*$ has been extended over $(-\infty,a)$ by the definition $\alpha(x) = \alpha(a)$ for $x \in (-\infty,a)$.

To avoid some trivial cases we shall assume in this chapte: that the entire function $f(z)$ is not a polynomial.

6.2. We first obtain the coefficient characterizations of generalized (α,α)-order of an entire function in this section. Further, a theorem connecting generalized (α,α)-order and generalized lower (α,α)-order of an entire function with its maximum term and central index is proved in this section.

THEOREM 6.2.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having generalized* (α,α)-*order* $\rho_\infty(\alpha,\alpha,f)$, $1 \leq \rho_\infty(\alpha,\alpha,f) \leq \infty$. *Then*

$$\rho_\infty(\alpha,\alpha,f) = Q(U)$$

where

$$U = \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha\left(\frac{1}{\lambda_n}\log|a_n|^{-1}\right)}. \tag{6.2.1}$$

PROOF. We first note that, since $\log M(r,f)$ is a convex function of $\log r$, we have $\rho_\infty(\alpha,\alpha,f) \geq 1$.

If $\alpha(x) \in \Omega$, it follows from a result of Šeremeta [93], i.e., (1.3.5), that $\rho_\infty(\alpha,\alpha,f) \geq U$ and hence $\rho_\infty(\alpha,\alpha,f) \geq Q(U)$.

Next, let $\alpha(x) \in \bar{\Omega}$ and $\rho_\infty(\alpha,\alpha,f) < \infty$. Then, by (6.1.1), given $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that

$$\log M(r,f) < \tilde{F}(\log r, \bar{\rho})$$

for $r > r_0$, where $\bar{\rho} = \rho_\infty(\alpha,\alpha,f) + \varepsilon$. Thus, using Cauchy's inequality, for all $r > r_0$ and all n, we have

$$\log |a_n| < \tilde{F}(\log r, \bar{\rho}) - \lambda_n \log r. \tag{6.2.2}$$

We choose $r = r(n)$ to be the unique root of the equation

$$\lambda_n = \frac{\bar{\rho}}{\log r} \tilde{F}(\log r, \bar{\rho}). \tag{6.2.3}$$

For $n > n_0$, we have $r = r(n) > r_0$. By (6.2.2) and (6.2.3), for $n > n_0$, we have

$$\log |a_n|^{-1} > \lambda_n \log r - \tilde{F}(\log r, \bar{\rho}) = \frac{\bar{\rho}-1}{\bar{\rho}} \lambda_n \log r$$

and so

$$\alpha(\log r) < \alpha\left(\frac{\bar{\rho}}{\bar{\rho}-1} \frac{1}{\lambda_n} \log |a_n|^{-1}\right) \tag{6.2.4}$$

where $r = r(n)$ is given by (6.2.3). Since $\alpha(x) \in \bar{\Omega}$, we have

$$\alpha(\log r) \sim \frac{1}{\bar{\rho}-1} \alpha(\lambda_n) \tag{6.2.5}$$

as $n \to \infty$, for $r = r(n)$ satisfying (6.2.3). Thus, (6.2.4) and (6.2.5) give that $\bar{\rho} = \rho_\infty(\alpha,\alpha,f) + \varepsilon \geq Q(U)$ and so, letting $\varepsilon \to 0$, we have $\rho_\infty(\alpha,\alpha,f) \geq Q(U)$.

We proved above that $\rho_\infty(\alpha,\alpha,f) \geq Q(U)$. To prove that $\rho_\infty(\alpha,\alpha,f) \leq Q(U)$ we assume that $U < \infty$, since there is nothing to prove if $U = \infty$. Then, by (6.2.1), given $\varepsilon > 0$, there exists $n_o' = n_o'(\varepsilon)$ such that

$$|a_n| < \exp(-\lambda_n \tilde{F}(\lambda_n, 1/\bar{U}))$$

for $n > n_o'$, where $\bar{U} = U+\varepsilon$. Now

$$M(r,f) \leq \sum_{n=0}^{\infty} |a_n| r^{\lambda_n} = \sum_{n=0}^{n_o'} |a_n| r^{\lambda_n} + \sum_{n=n_o'+1}^{s} |a_n| r^{\lambda_n} + \sum_{n=s+1}^{\infty} |a_n| r^{\lambda_n}$$

where s is chosen such that

$$\lambda_s \leq \tilde{F}(\log 2r, \bar{U}) < \lambda_{s+1}$$

and so

$$M(r,f) \leq A(n_o') + \exp(\tilde{F}(\log 2r,\bar{U})\log r) \sum_{n=0}^{\infty} \exp(-\lambda_n \tilde{F}(\lambda_n,1/\bar{U})) + \sum_{n=0}^{\infty} 2^{-n}$$

where $A(n_o')$ is a polynomial of degree atmost $\lambda_{n_o'}$. Since both the series in the above expression are convergent, for all sufficiently large values of r, we have

$$(1+o(1)) \log M(r,f) \leq \tilde{F}(\log 2r,\bar{U}) \log r. \tag{6.2.6}$$

Now, let $\alpha(x) \in \Omega$. To prove $\rho_\infty(\alpha,\alpha,f) \leq \max(1,U)$, first let $U < 1$ and let $\varepsilon > 0$ be chosen such that $U+\varepsilon = \bar{U} < 1$. Then, by (6.2.6), since $\bar{U} < 1$, we have

$$(1+o(1)) \log M(r,f) \leq (\log 2r)\log r = (\log r)^2 (1+o(1))$$

which gives that $\rho_\infty(\alpha,\alpha,f) \leq 1$. Next, if $U \geq 1$ then $U+\varepsilon = \bar{U} > 1$ and so, by (6.2.6), we have

$$(1+o(1)) \log M(r,f) \leq (\tilde{F}(\log 2r, \bar{U}))^2$$

which gives

$$\begin{aligned}\alpha((1+o(1)) \log M(r,f)) &\leq \alpha((\tilde{F}(\log 2r, \bar{U}))^2) \\ &\sim \alpha(\tilde{F}(\log 2r, \bar{U})) \\ &= \bar{U}\, \alpha(\log 2r),\end{aligned}$$

since for $\alpha(x) \in \Omega$ we have $\alpha(x^2) \sim \alpha(x)$ as $x \to \infty$. Dividing the above inequality by $\alpha(\log r)$ and passing to limits we get $\rho_\infty(\alpha,\alpha,f) \leq \bar{U} = U+\varepsilon$, and so letting $\varepsilon \to 0$ we have

$$(6.2.7) \qquad \rho_\infty(\alpha,\alpha,f) \leq U.$$

Now, let $\alpha(x) \in \bar{\Omega}$. Then by (6.2.6), we obtain

$$\begin{aligned}\alpha((1+o(1)) \log M(r,f)) &\leq \alpha(\tilde{F}(\log 2r, \bar{U}) \log r) \\ &= \alpha(\tilde{F}(\log 2r, \bar{U})) + \log \log r \left.\frac{d\alpha(x)}{d \log x}\right|_{x=x^*(r)}\end{aligned}$$

where $\tilde{F}(\log 2r, \bar{U}) < x^*(r) < \tilde{F}(\log 2r, \bar{U}) \log r$. This easily gives that $\rho_\infty(\alpha,\alpha,f) \leq (U+\varepsilon)+1$ and so letting $\varepsilon \to 0$ we get

$$(6.2.8) \qquad \rho_\infty(\alpha,\alpha,f) \leq 1+U.$$

From (6.2.7) and (6.2.8) we get $\rho_\infty(\alpha,\alpha,f) \leq Q(U)$ and hence $\rho_\infty(\alpha,\alpha,f) = Q(U)$.

This proves the theorem.

<u>REMARK</u>. With $\alpha(x) = \log_p x$, $p \geq 1$, a result in [45] follows from the above theorem.

For proving our next theorem we need the following lemmas.

LEMMA 6.2.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function and let* $\alpha(x)$ *belong to* Λ_*. *Then*

$$(6.2.9)\qquad \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha\left(\frac{1}{\lambda_n-\lambda_{n-1}} \log |a_{n-1}/a_n|\right)} \geq \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha\left(\frac{1}{\lambda_n} \log |a_n|^{-1}\right)}$$

PROOF. Clearly, the assertion of the lemma is true if $(\log |a_{n-1}/a_n|)/(\lambda_n-\lambda_{n-1}) \leq a$ for infinitely many values of n, where a is such that $\alpha(x)$ is defined on $[a,\infty)$; since in that case the limit superior on the left hand side of (6.2.9) will be ∞. Thus, let $(\log |a_{n-1}/a_n|)/(\lambda_n-\lambda_{n-1}) > a$ for all sufficiently large values of n. Let the limit superior on the left hand side of (6.2.9) be denoted by U_o. Clearly $0 \leq U_o \leq \infty$. Since (6.2.9) is trivial for $U_o = \infty$, let $U_o < \infty$. Then, given $\varepsilon > 0$, there exists $n_o = n_o(\varepsilon)$ such that

$$\log |a_{n-1}/a_n| > (\lambda_n-\lambda_{n-1})\ \tilde{F}(\lambda_n, 1/\bar{U}_o)$$

for $n > n_o$, where $\bar{U}_o = U_o+\varepsilon$. Writing the above inequality for $n = n_o+1, n_o+2, \ldots, k$ and adding, we get

$$\log |a_{n_o}/a_k| > (\lambda_{n_o+1}-\lambda_{n_o})\ \tilde{F}(\lambda_{n_o+1}, 1/\bar{U}_o)+\ldots+(\lambda_k-\lambda_{k-1})\tilde{F}(\lambda_k, 1/\bar{U}_o)$$

$$= \lambda_k\ \tilde{F}(\lambda_k, 1/\bar{U}_o) - \int_{\lambda_{n_o+1}}^{\lambda_k} n(t)\, d(\tilde{F}(t, 1/\bar{U}_o)) - \lambda_{n_o}\ \tilde{F}(\lambda_{n_o+1}, 1/\bar{U}_o)$$

where $n(t) = \lambda_m$ for $\lambda_m < t \leq \lambda_{m+1}$. Thus

$$\log|a_k|^{-1} > \log|a_{n_o}|^{-1} + \lambda_k \tilde{F}(\lambda_k, 1/\bar{U}_o) - \int_{\lambda_{n_o+1}}^{\lambda_k} t\, d(\tilde{F}(t,1/\bar{U}_o))$$
$$- \lambda_{n_o} \tilde{F}(\lambda_{n_o+1}, 1/\bar{U}_o)$$

$$= \log|a_{n_o}|^{-1} + \int_{\lambda_{n_o+1}}^{\lambda_k/2} \tilde{F}(t,1/\bar{U}_o)dt + \int_{\lambda_k/2}^{\lambda_k} \tilde{F}(t,1/\bar{U}_o)\, dt$$

$$+ (\lambda_{n_o+1} - \lambda_{n_o})\ \tilde{F}(\lambda_{n_o+1}, 1/\bar{U}_o)$$

$$> \log|a_{n_o}|^{-1} + (\lambda_k/2 - \lambda_{n_o+1})\tilde{F}(\lambda_{n_o+1}, 1/\bar{U}_o) +$$

$$+ (\lambda_k/2)\tilde{F}(\lambda_k/2, 1/\bar{U}_o) + (\lambda_{n_o+1} - \lambda_{n_o})\ \tilde{F}(\lambda_{n_o+1}, 1/\bar{U}_o)$$

$$= (1+o(1))\ (\lambda_k/2)\ \tilde{F}(\lambda_k/2,\ 1/\bar{U}_o).$$

Since $\alpha(x) \in \Lambda_*$, the lemma follows easily from the above inequality.

<u>LEMMA</u> 6.2.2. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ <u>be an entire function such that</u> $\psi_o(n) = (\log|a_n/a_{n+1}|)/(\lambda_{n+1}-\lambda_n)$ <u>is a nondecreasing function of</u> n <u>for</u> $n \geq n_o$. <u>Then, for</u> $\alpha(x) \in \Lambda_*$, <u>we have</u>

$$\limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha(\frac{1}{\lambda_n - \lambda_{n-1}} \log|a_{n-1}/a_n|)} \leq \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha(\frac{1}{\lambda_n}\log|a_n|^{-1})}.$$

<u>PROOF</u>. Since $\psi_o(n)$ is a nondecreasing function of n for $n \geq n_o$, we have

$$\log|a_{n_o}/a_n| = \log|a_{n_o}/a_{n_o+1}| + \dots + \log|a_{n-1}/a_n|$$

$$= (\lambda_{n_o+1} - \lambda_{n_o})\ \Psi_o(n_o) + \dots + (\lambda_n - \lambda_{n-1})\ \Psi_o(n-1)$$

or

$$(1+o(1))\,\frac{1}{\lambda_n}\log|a_n|^{-1} \le \Psi_o(n-1).$$

The lemma follows from the above inequality, since $\alpha(x)\ \varepsilon\ \Lambda_*$.

Our next theorem gives a characterization of generalized (α,α)-order of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, in terms of the ratio of consecutive coefficients a_n. The characterization holds for a subclass of entire functions.

THEOREM 6.2.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function having generalized (α,α)-order $\rho_\infty(\alpha,\alpha,f)$ $(1 \le \rho_\infty(\alpha,\alpha,f) \le \infty)$. Assume that $\psi_o(n) = (\log|a_n/a_{n+1}|)/(\lambda_{n+1}-\lambda_n)$ is ultimately a nondecreasing function of n. Then*

$$\rho_\infty(\alpha,\alpha,f) = Q(U_o)$$

where

$$U_o = \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha\left(\frac{1}{\lambda_n-\lambda_{n-1}}\log|a_{n-1}/a_n|\right)}.$$

PROOF. Since $\psi_o(n)$ is ultimately a nondecreasing function of n, it follows from Lemmas 6.2.1 and 6.2.2 that $U_o = U$, where U is given by (6.2.1). The theorem now follows from Theorem 6.2.1.

REMARK. Taking $\alpha(x) = \log_p x$, $p \ge 1$, a result in [45] follows from the above theorem.

Our next theorem correlates the generalized (α,α)-order and generalized lower (α,α)-order of an entire function with its maximum term and the central index.

THEOREM 6.2.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function with generalized* (α,α)-*order* $\rho_\infty(\alpha,\alpha,f)$ *and generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$. *Then*

$$\rho_\infty(\alpha,\alpha,f) = Q(\phi_1) = \theta_1 \tag{6.2.10}$$

and

$$\lambda_\infty(\alpha,\alpha,f) = Q(\phi_2) = \theta_2 \tag{6.2.11}$$

where

$$\begin{matrix}\phi_1 \\ \phi_2\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\nu(r))}{\alpha(\log r)} \tag{6.2.12}$$

and

$$\begin{matrix}\theta_1 \\ \theta_2\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log \mu(r))}{\alpha(\log r)} . \tag{6.2.13}$$

PROOF. We shall prove the theorem in several parts.

(i) Since $\log M(r,f)$ is a convex function of $\log r$ we have $\rho_\infty(\alpha,\alpha,f) \geq \lambda_\infty(\alpha,\alpha,f) \geq 1$.

(ii) Since $\alpha(x) \in \Lambda_*$, $\rho_\infty(\alpha,\alpha,f) = \theta_1$ and $\lambda_\infty(\alpha,\alpha,f) = \theta_2$ follows from the result (1.4.8) of Shah [102].

(iii) Let $\alpha(x) \in \Omega$. We first prove that $\theta_2 \geq \max(1,\phi_2)$. By (1.4.1), we have

$$\log \mu(er) > \nu(r) \tag{6.2.14}$$

and so $\theta_2 \geq \phi_2$. By parts (i) and (ii) of the proof we have $\theta_2 \geq 1$ and so $\theta_2 \geq \max(1,\phi_2)$.

To prove $\theta_2 \leq \max(1,\phi_2)$, we can assume that $\phi_2 < \infty$, since there is nothing to prove if $\phi_2 = \infty$. First, let $\phi_2 < 1$ and

$\phi_2 < \bar{\phi}_2 < 1$. Then, by (6.2.12), we have

$$\nu(r_n) < \tilde{F}(\log r_n, \bar{\phi}_2)$$

for a sequence $\{r_n\}$ such that $r_n \to \infty$ as $n \to \infty$. From (1.4.1) and the above inequality, since $\bar{\phi}_2 < 1$, we have

$$\begin{aligned}(1+o(1)) \log \mu(r_n) < \nu(r_n) \log r_n < \tilde{F}(\log r_n, \bar{\phi}_2) \log r_n \\ \leq (\log r_n)^2\end{aligned}$$

for the sequence $\{r_n\}$. Since $\alpha(x) \in \Omega$, the above relation gives that $\theta_2 \leq 1$. Next, let $1 \leq \phi_2 < \infty$. Then, given $\varepsilon > 0$, from (6.2.12), we have

$$\nu(r_n') < \tilde{F}(\log r_n', \phi_2+\varepsilon)$$

for a sequence $\{r_n'\}$, $r_n' \to \infty$ as $n \to \infty$. From (1.4.1) and the above relation, since $\phi_2+\varepsilon > 1$, we have

$$\begin{aligned}(1+o(1)) \log \mu(r_n') < \nu(r_n') \log r_n' < \tilde{F}(\log r_n', \phi_2+\varepsilon) \log r_n' \\ \leq (\tilde{F}(\log r_n', \phi_2+\varepsilon))^2\end{aligned}$$

for the sequence $\{r_n'\}$. This gives

$$\frac{\alpha(((1+o(1)) \log \mu(r_n'))^{1/2})}{\alpha(\log r_n')} \leq \phi_2+\varepsilon$$

for the sequence $\{r_n'\}$. Since $\alpha(x^2) \sim \alpha(x)$ for $\alpha(x) \in \Omega$ and $\varepsilon > 0$ is arbitrary, we now have $\theta_2 \leq \phi_2$. Hence $\theta_2 = \max(1, \phi_2)$.

(iv) Let $\alpha(x) \in \Omega$. Using (6.2.14) we get $\theta_1 \geq \phi_1$. By parts (i) and (ii) of the proof we have $\theta_1 \geq 1$ and so $\theta_1 \geq \max(1, \phi_1)$.

To prove $\theta_1 \le \max(1,\phi_1)$ we assume that $\phi_1 < \infty$ since otherwise there is nothing to prove. Then, given $\varepsilon > 0$, by (6.2.12), there exists $r_o = r_o(\varepsilon)$ such that for $r > r_o$, we have

$$\nu(r) < \tilde{F}(\log r,\ \phi_1+\varepsilon).$$

The above relation, in view of (1.4.1), for $r > r_o$, gives that

(6.2.15) $$(1+o(1))\log\mu(r) < \nu(r)\log r < \tilde{F}(\log r,\phi_1+\varepsilon)\log r.$$

If $\phi_1 < 1$, then, for some $\varepsilon > 0$, we have $\phi_1+\varepsilon < 1$ and so from (6.2.15), for $r > r_o$, we have

$$(1+o(1))\log\mu(r) < (\log r)^2.$$

Since, $\alpha(x^2) \sim \alpha(x)$ as $x \to \infty$, for $\alpha(x) \in \Omega$, the above relation gives that $\theta_1 \le 1$. If $\phi_1 \ge 1$, then (6.2.15) gives that

$$(1+o(1))\log\mu(r) \le (\tilde{F}(\log r,\phi_1+\varepsilon))^2$$

for $r > r_o$, since $\phi_1+\varepsilon > 1$. The above relation, since $\alpha(x) \in \Omega$ and $\varepsilon > 0$ is arbitrary, gives that $\theta_1 \le \phi_1$. Hence $\theta_1 = \max(1,\phi_1)$.

(v) Now, let $\alpha(x) \in \bar{\Omega}$. We first prove that $\theta_2 \le 1+\phi_2$. By (1.4.1), we have

$$\alpha((1+o(1))\log\mu(r)) < \alpha(\nu(r)\log r)$$

$$= \alpha(\nu(r)) + \log\log r \left.\frac{d\alpha(x)}{d\log x}\right|_{x=x^*(r)}$$

where $\nu(r) < x^*(r) < \nu(r)\log r$. Since $\alpha(x) \in \bar{\Omega}$, this gives that $\theta_2 \le 1+\phi_2$.

Again, using (1.4.1), we get

$$\log \mu(r^2) = \log \mu(r) + \int_r^{r^2} \frac{\nu(t)}{t}\, dt > \nu(r) \log r$$

for all sufficiently large values of r. Proceeding as above we get $\theta_2 \geq 1+\phi_2$. Hence $\theta_2 = 1+\phi_2$.

(vi) Proof of $\theta_1 = 1+\phi_1$, when $\alpha(x) \in \bar{\Omega}$, is similar to part (v) above.

Theorem follows from parts (i) to (vi) above.

REMARK. Taking $\alpha(x) = \log_p x$, $p \geq 1$, some results of Juneja, Kapoor and Bajpai [45] and Shah and Ishaq [103] follow from the above theorem.

6.3. In this section, we first obtain coefficient characterizations of generalized lower (α,α)-order of an entire function. We also obtain here a necessary and sufficient condition for an entire function to be of generalized regular (α,α)-growth.

We need the following lemmas.

LEMMA 6.3.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having the generalized lower* (α,α)*-order* $\lambda_\infty(\alpha,\alpha,f)$ $(1 \leq \lambda_\infty(\alpha,\alpha,f) \leq \infty)$. *Let* $\{n_k\}$ *be an increasing sequence of positive integers. Then*

$$(6.3.1) \qquad \lambda_\infty(\alpha,\alpha,f) \geq Q_{\chi(\{n_k\})}(V(\{n_k\}))$$

where

$$(6.3.2) \qquad \chi(\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\lambda_{n_k})}$$

and

$$(6.3.3) \qquad V(\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\frac{1}{\lambda_{n_k}} \log |a_{n_k}|^{-1})} .$$

PROOF. We note that $0 \leq V(\{n_k\}) \leq \infty$. Obviously, (6.3.1) holds for $V(\{n_k\}) = 0$, since $\lambda_\infty(\alpha,\alpha,f) \geq 1$. Now, let $0 < V(\{n_k\}) < \infty$ and let $V(\{n_k\}) > \varepsilon > 0$. Then, for $k > k_o = k_o(\varepsilon)$, we have

$$|a_{n_k}| > \exp(-\lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V})),$$

where $\bar{V} = V(\{n_k\}) - \varepsilon$. The above inequality, in view of Cauchy's inequality, for $k > k_o$, gives that

$$(6.3.4) \qquad \log M(r,f) \geq \log |a_{n_k}| + \lambda_{n_k} \log r$$

$$\geq -\lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V}) + \lambda_{n_k} \log r.$$

If $\alpha(x) \in \Omega$, for $k > k_o$, we choose

$$r_k = \exp(1 + \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V})).$$

Then, for $r_k \leq r \leq r_{k+1}$, using (6.3.4), we get

$$\log M(r,f) \geq -\lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V}) + \lambda_{n_k} \log r_k = \lambda_{n_k}$$

$$= \tilde{F}(\log r_{k+1} - 1, \bar{V}) \geq \tilde{F}(\log r - 1, \bar{V}).$$

The above inequality, since $\alpha(x) \in \Omega$ and $\varepsilon > 0$ is arbitrary, gives $\lambda_\infty(\alpha,\alpha,f) \geq V(\{n_k\})$ and so $\lambda_\infty(\alpha,\alpha,f) \geq \max(1, V(\{n_k\}))$, because $\lambda_\infty(\alpha,\alpha,f) \geq 1$.

If $\alpha(x) \in \bar{\Omega}$, for $k > k_o$, we take

$$(6.3.5) \qquad r_k = \exp\left(2\tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V})\right).$$

Then, for $r_k \leq r \leq r_{k+1}$, from (6.3.4), we get

$$\log M(r,f) \geq -\lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V}) + \lambda_{n_k} \log r_k$$

$$= \lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V})$$

and so

$$\alpha(\log M(r,f)) \geq \alpha(\lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V}))$$

$$= \frac{1}{\bar{V}} \alpha(\lambda_{n_{k-1}}) + \log \lambda_{n_k} \left.\frac{d\alpha(x)}{d \log x}\right|_{x=x^*(k)}$$

where $\tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V}) < x^*(k) < \lambda_{n_k} \tilde{F}(\lambda_{n_{k-1}}, 1/\bar{V})$. For $r_k \leq r \leq r_{k+1}$, this gives

$$\frac{\alpha(\log M(r,f))}{\alpha(\log r/2)} \geq \frac{1}{\bar{V}} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\log r_{k+1}/2)} + \frac{\log \lambda_{n_k}}{\alpha(\log r_{k+1}/2)} \left.\frac{d\alpha(x)}{d \log x}\right|_{x=x^*(k)}.$$

Since $\alpha(x) \in \bar{\Omega}$ and $\varepsilon > 0$ is arbitrary, the above relation, on using (6.3.5), gives that $\lambda_\infty(\alpha,\alpha,f) \geq \chi(\{n_k\}) + V(\{n_k\})$.

If $V(\{n_k\}) = \infty$, the above arguments with an arbitrarily large number in place of $\bar{V}$ give that $\lambda_\infty(\alpha,\alpha,f)$ is also ∞.

This proves the lemma.

LEMMA 6.3.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function with generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$, $1 \leq \lambda_\infty(\alpha,\alpha,f) \leq \infty$. *Let* $\{n_k\}$ *be an increasing sequence of positive integers. Then*

$$\lambda_\infty(\alpha,\alpha,f) \geq Q_{\chi(\{n_k\})}(V_0(\{n_k\})),$$

where $\chi(\{n_k\})$ *is defined by* (6.3.2) *and*

$$(6.3.6) \qquad V_o(\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\frac{1}{\lambda_{n_k} - \lambda_{n_{k-1}}} \log |a_{n_{k-1}}/a_{n_k}|)} .$$

PROOF. Let $V(\{n_k\})$ be given by (6.3.3). It can be easily verified that (see [5, Lemma 7])

$$(6.3.7) \qquad V(\{n_k\}) \geq V_o(\{n_k\}),$$

for any increasing sequence $\{n_k\}$ of positive integers. The lemma now follows from Lemma 6.3.1 and (6.3.7).

LEMMA 6.3.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function and let* $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *be a nondecreasing function of* n *for* $n > n_o$. *Then*

$$(6.3.8) \qquad V^o \geq \phi_2$$

and

$$(6.3.9) \qquad V^* \geq \phi_2$$

where ϕ_2, *for* $\alpha(x) \in \Lambda_*$, *is defined by* (6.2.12),

$$(6.3.10) \qquad V^o = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\frac{1}{\lambda_n} \log |a_n|^{-1})}$$

and

$$(6.3.11) \qquad V^* = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\frac{1}{\lambda_n - \lambda_{n-1}} \log |a_{n-1}/a_n|)} .$$

PROOF. Since $\psi(n)$ is nondecreasing for $n > n_o$, it follows that $\psi(n) > \psi(n-1)$ for infinitely many values of n because if it is

not so, then $\psi(n) = \Psi(n+1) = \ldots$ ad infinitum, for $n > n_o$(say) and so the radius of convergence of $\sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is finite which is a contradiction. Further $\psi(n) \to \infty$ as $n \to \infty$.

When $\Psi(n) > \Psi(n-1)$ the term $a_n z^{\lambda_n}$ becomes maximum term and we have

$$\mu(r) = |a_n|\, r^{\lambda_n}, \quad \nu(r) = \lambda_n \text{ for } \Psi(n-1) \leq r < \Psi(n).$$

Now, let $\infty > \phi_2 > 0$ and $\phi_2 > \varepsilon > 0$. Then for all $r > r_o = r_o(\varepsilon)$, by (6.2.12), we have

$$\nu(r) > \tilde{F}(\log r, \phi_2-\varepsilon).$$

Let $a_{n_1} z^{\lambda_{n_1}}$ and $a_{n_2} z^{\lambda_{n_2}}$ $(n_1 > n_o,\ \Psi(n_1-1) > r_o)$ be two consecutive maximum terms so that $n_1 \leq n_2-1$. Let $n_1 < n \leq n_2$. Since $a_{n_1} z^{\lambda_{n_1}}$ is maximum term we have $\nu(r) = \lambda_{n_1}$ for $\Psi(n_1-1) \leq r < \Psi(n_1)$. Hence, for any r in this interval

$$\lambda_{n_1} = \nu(r) > \tilde{F}(\log r, \phi_2-\varepsilon).$$

Further, since

$$\psi(n_1) = \Psi(n_1+1) = \ldots = \Psi(n-1),$$

we have

$$(6.3.12) \qquad \lambda_{n-1} \geq \lambda_{n_1} > \tilde{F}(\log(\Psi(n-1)-c_o),\ \phi_2-\varepsilon),$$

where $c_o = \min\left[1, (\psi(n_1)-\Psi(n_1-1))/2\right]$. Now, since $\psi(n)$ is nondecreasing

$$\log|a_{n_o}/a_n| = \log|a_{n_o}/a_{n_o+1}|+\dots+\log|a_{n-1}/a_n|$$
$$\leq (\lambda_n - \lambda_{n_o}) \log \Psi(n-1)$$
$$\leq \lambda_n \log \psi(n-1)$$

and so

$$(1+o(1)) \frac{1}{\lambda_n} \log |a_n|^{-1} \leq \log \Psi(n-1).$$

The above relation, in view of (6.3.12), since $\alpha(x) \varepsilon \Lambda_*$, gives that

$$\liminf_{n\to\infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\frac{1}{\lambda_n} \log |a_n|^{-1})} \geq \phi_2-\varepsilon.$$

Letting $\varepsilon \to 0$, (6.3.8) follows from the above relation for $0 < \phi_2 < \infty$. Obviously, (6.3.8) holds for $\phi_2 = 0$. For $\phi_2 = \infty$, the above analysis with an arbitrarily large number in place of $\phi_2-\varepsilon$ gives that $V^o = \infty$.

Next, putting the value of $\psi(n-1)$ in (6.3.12) and using the fact that $\alpha(x) \varepsilon \Lambda_*$, we get

$$\liminf_{n\to\infty} \frac{\alpha(\lambda_{n-1})}{\alpha(\frac{1}{\lambda_n-\lambda_{n-1}} \log |a_{n-1}/a_n|)} \geq \phi_2-\varepsilon.$$

Letting $\varepsilon \to 0$, (6.3.9) follows from the above inequality for $0 < \phi_2 < \infty$. Obviously, (6.3.9) holds for $\phi_2 = 0$. For $\phi_2 = \infty$, the above arguments with an arbitrarily large number in place of $\phi_2-\varepsilon$ give that $V^* = \infty$.

The lemma is thus proved.

We now have the following coefficient characterization of generalized lower (α,α)-order $\lambda_\infty(\alpha,\alpha,f)$ of an entire function $f(z)$.

THEOREM 6.3.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$, $1 \le \lambda_\infty(\alpha,\alpha,f) \le \infty$. *Assume that* $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *is ultimately a nondecreasing function of* n. *Then*

(i) *If* $\alpha(x) \in \Omega$, *we have*

$$\lambda_\infty(\alpha,\alpha,f) = Q(V^{o}) = Q(V^{*}) \tag{6.3.13}$$

where V^{o} *and* V^{*} *are defined by* (6.3.10) *and* (6.3.11), *respectively.*

(ii) *If* $\alpha(x) \in \bar{\Omega}$, *then* (6.3.13) *holds under the additional restriction that* $\alpha(\lambda_n) \sim \alpha(\lambda_{n+1})$ *as* $n \to \infty$.

PROOF. The theorem is immediate in view of Lemmas 6.3.1, 6.3.2 and 6.3.3 and so we omit the details.

Our next theorem gives a coefficient characterization of generalized lower (α,α)-order of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ which does not require the nondecreasing nature of the function $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$.

THEOREM 6.3.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function having generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$ $(1 \le \lambda_\infty(\alpha,\alpha,f) \le \infty)$. *Then*

(i) *If* $\alpha(x) \in \Omega$, *we have*

$$(6.3.14) \qquad \lambda_\infty(\alpha,\alpha,f) = \max_{\{n_k\}} \left[Q_{\chi(\{n_k\})} (V(\{n_k\})) \right]$$

<u>and</u>

$$(6.3.15) \qquad \lambda_\infty(\alpha,\alpha,f) = \max_{\{n_k\}} \left[Q_{\chi(\{n_k\})} (V_o(\{n_k\})) \right]$$

<u>where</u> $\chi(\{n_k\})$, $V(\{n_k\})$ <u>and</u> $V_o(\{n_k\})$ <u>are defined by</u> (6.3.2), (6.3.3) <u>and</u> (6.3.6), <u>respectively, and the maximum in</u> (6.3.14) <u>and</u> (6.3.15) <u>is taken over all increasing sequences</u> $\{n_k\}$ <u>of positive integers</u>.

(ii) <u>Further, if</u> $\{\lambda_{n_m}\}$ <u>is the sequence of the principal indices of</u> $f(z)$ <u>such that</u> $\alpha(\lambda_{n_m}) \sim \alpha(\lambda_{n_{m+1}})$ <u>as</u> $m \to \infty$, <u>then</u> (6.3.14) <u>and</u> (6.3.15) <u>hold for</u> $\alpha(x) \in \bar{\Omega}$ <u>also</u>.

<u>PROOF</u>. Consider the function $g(z) = \sum_{m=0}^{\infty} a_{n_m} z^{\lambda_{n_m}}$, where $\{\lambda_{n_m}\}$ is the sequence of the principal indices of $f(z)$. Then $g(z)$ is also an entire function. Further, for any z, $f(z)$ and $g(z)$ have the same maximum term and so, by Theorem 6.2.3, $f(z)$ and $g(z)$ have the same generalized lower (α,α)-order. Thus, the generalized lower (α,α)-order of $g(z)$ is $\lambda_\infty(\alpha,\alpha,f)$. Further, $\psi(n_m) = |a_{n_m}/a_{n_{m+1}}|^{1/(\lambda_{n_{m+1}}-\lambda_{n_m})}$ is a strictly increasing function of m. Now, since $g(z)$ satisfies the hypothesis of Theorem 6.3.1, we have

$$(6.3.16) \qquad \lambda_\infty(\alpha,\alpha,f) = Q_{\chi(\{n_m\})} (V(\{n_m\})),$$

$$(6.3.17) \qquad \lambda_\infty(\alpha,\alpha,f) = Q_{\chi(\{n_m\})} (V_o(\{n_m\})).$$

On the other hand, applying Lemmas 6.3.1 and 6.3.2 to the function f(z), we get

$$(6.3.18) \qquad \lambda_\infty(\alpha,\alpha,f) \geq \max_{\{n_k\}} \left[Q_{\chi(\{n_k\})} (V(\{n_k\})) \right]$$

and

$$(6.3.19) \qquad \lambda_\infty(\alpha,\alpha,f) \geq \max_{\{n_k\}} \left[Q_{\chi(\{n_k\})} (V_o(\{n_k\})) \right] .$$

Combining (6.3.16) and (6.3.18) we get (6.3.14), while (6.3.17) and (6.3.19) give (6.3.15). This proves the theorem.

<u>REMARK</u>. With $\alpha(x) = \log_p x$, $p \geq 1$, some results in [45] follow from Theorems 6.3.1 and 6.3.2.

We now prove

<u>THEOREM</u> 6.3.3. <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ <u>be an entire function. For</u> $\alpha(x) \in \Lambda_*$, <u>let</u> ϕ_1 <u>and</u> ϕ_2 <u>be defined by</u> (6.2.12). <u>Then</u>

$$\phi_2 \leq \phi_1 \liminf_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})} .$$

<u>PROOF</u>. Let $\liminf_{n \to \infty} \alpha(\lambda_n)/\alpha(\lambda_{n+1}) = b$. If $b' > b$, we have

$$\alpha(\lambda_{n_t}) < b' \, \alpha(\lambda_{n_t+1})$$

for a sequence $\{n_t\}_{t=0}^{\infty}$ such that $n_t \to \infty$ as $t \to \infty$. Let r_t be a value of r at which $\nu(r)$ jumps from a value less than or equal to λ_{n_t} to a value greater than or equal to λ_{n_t+1}. Then

$$\alpha(\nu(r_t-0)) \leq \alpha(\lambda_{n_t}) < b' \, \alpha(\lambda_{n_t+1}) \leq b' \, \alpha(\nu(r_t+0))$$

and so, since $\alpha(x) \in \Lambda_*$, we have

$$\phi_2 \leq \limsup_{t \to \infty} \frac{\alpha(\nu(r_t - 0))}{\alpha(\log r_t)} \leq b' \limsup_{t \to \infty} \frac{\alpha(\nu(r_t + 0))}{\alpha(\log r_t)} \leq b' \phi_1 .$$

Since the above inequality is true for every $b' > b$ we get $\phi_2 \leq b\, \phi_1$. This proves the theorem.

COROLLARY 6.3.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function with generalized* (α,α)-*order* $\rho_\infty(\alpha,\alpha,f)$ *and generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$. *Then*

(i) *If* $\alpha(x) \in \Omega$ *and* $\lambda_\infty(\alpha,\alpha,f) > 1$, *we have*

$$\lambda_\infty(\alpha,\alpha,f) \leq \rho_\infty(\alpha,\alpha,f) \liminf_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})} .$$

(ii) *If* $\alpha(x) \in \bar{\Omega}$, *we have*

$$\lambda_\infty(\alpha,\alpha,f) - 1 \leq (\rho_\infty(\alpha,\alpha,f) - 1) \liminf_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})} .$$

The corollary is immediate in view of Theorem 6.2.3 and Theorem 6.3.3 and so we omit the proof.

COROLLARY 6.3.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function. Then a necessary and sufficient condition that* $f(z)$ *should be of generalized regular* (α,α)-*growth with* $1 < \rho_\infty(\alpha,\alpha,f) < \infty$ *is that for every* $\varepsilon > 0$ *there exists* $n_o = n_o(\varepsilon)$ *such that for all* $n > n_o$ *we have*

$$|a_n| < \exp\left[-\lambda_n \tilde{F}(\lambda_n, 1/(\rho^* + \varepsilon)) \right] \tag{6.3.20}$$

where $\rho^* = \rho_\infty(\alpha,\alpha,f)$ *if* $\alpha(x) \in \Omega$ *and* $\rho^* = \rho_\infty(\alpha,\alpha,f) - 1$ *if* $\alpha(x) \in \bar{\Omega}$, *and that there exists a strictly increasing sequence* $\{n_p\}$ *of positive integers such that*

(6.3.21) $\alpha(\lambda_{n_p}) \sim \alpha(\lambda_{n_{p+1}})$ as $p \to \infty$

and

(6.3.22) $Q(U_*) = \rho_\infty(\alpha,\alpha,f)$

where

$$U_* = \lim_{p \to \infty} \frac{\alpha(\lambda_{n_p})}{\alpha(\frac{1}{\lambda_{n_p}} \log |a_{n_p}|^{-1})} .$$

PROOF. We first prove the necessity part. Thus, let $f(z)$ be an entire function of generalized regular (α,α)-growth with $1 < \rho_\infty(\alpha,\alpha,f) < \infty$. Then the coefficients a_n of $f(z)$ satisfy (6.3.20) follows from Theorem 6.2.1. Now, consider the function $g(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}$, where $\{\lambda_{n_p}\}$ is the sequence of the principal indices of $f(z)$. Then $g(z)$ is an entire function and, for any z, $f(z)$ and $g(z)$ have the same maximum term. Hence, by Theorem 6.2.3, generalized (α,α)-order and generalized lower (α,α)-order of $f(z)$ and $g(z)$ are the same. This gives that $g(z)$ is of generalized regular (α,α)-growth with generalized (α,α)-order $\rho_\infty(\alpha,\alpha,f)$. Since $1 < \rho_\infty(\alpha,\alpha,f) < \infty$, applying Corollary 6.3.1 to $g(z)$, we get $\alpha(\lambda_{n_p}) \sim \alpha(\lambda_{n_{p+1}})$ as $p \to \infty$. Further, since $\Psi(n_p) = |a_{n_p}/a_{n_{p+1}}|^{1/(\lambda_{n_{p+1}}-\lambda_{n_p})}$ is a strictly increasing function of p, (6.3.22) follows from Theorems 6.2.1 and 6.3.1.

Sufficiency part follows from Theorem 6.2.1 and Lemma 6.3.1. This proves the corollary.

REMARKS (i) Taking $\alpha(x) = \log_p x$, $p \geq 1$, a results in [45] follows from the above theorem.

(ii) For the familiar order and lower order of an entire function, given by (1.2.2) and (1.2.4), results analogous to Theorem 6.3.3 and Corollary 6.3.1 are due to Whittaker [131].

6.4. In this section we prove a decomposition theorem for entire functions of generalized irregular (α,α)-growth.

THEOREM 6.4.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be an entire function of generalized irregular* (α,α)*-growth and let* $\lambda_\infty(\alpha,\alpha,f) < u < \rho_\infty(\alpha,\alpha,f)$. *Then* $f(z)$ *is of the form* $g_u(z) + h_u(z)$, *where* $g_u(z)$ *is an entire function with generalized* (α,α)*-order less than or equal to* u *and*

$$h_u(z) = \sum_{p=0}^{\infty} a_{m_p} z^{\lambda_{m_p}}$$

satisfies

$$\lambda_\infty(\alpha,\alpha,f) \geq u \liminf_{p\to\infty} \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})}.$$

PROOF. Let $g_u(z) = \Sigma' a_n z^{\lambda_n}$, where Σ' denotes the summation over n for which

$$|a_n| \leq \exp(-\lambda_n \tilde{F}(\lambda_n, 1/u^*)),$$

where $u^* = u$ if $\alpha(x) \in \Omega$ and $u^* = u-1$ if $\alpha(x) \in \bar{\Omega}$. Then, by Theorem 6.2.1, generalized (α,α)-order of $g_u(z)$ is less than or equal to u. Now, let

$$h_u(z) = f(z) - g_u(z) = \sum_{p=0}^{\infty} a_{m_p} z^{\lambda_{m_p}}.$$

Then

(6.4.1) $\quad |a_{m_p}| > \exp\left(-\lambda_{m_p}\tilde{F}(\lambda_{m_p}, 1/u^*)\right).$

If $\alpha(x) \in \Omega$, we take $r_p = \exp\left(1+\tilde{F}(\lambda_{m_p}, 1/u)\right)$. Then, for $r_p \leq r \leq r_{p+1}$, by Cauchy's inequality and (6.4.1), we have

$$\log M(r,f) \geq \log |a_{m_p}| + \lambda_{m_p} \log r \geq \log |a_{m_p}| + \lambda_{m_p} \log r_p$$
$$> \lambda_{m_p}$$

and so

$$\frac{\alpha(\log M(r,f))}{\alpha(\log r)} > \frac{\alpha(\lambda_{m_p})}{\alpha(\log r_{p+1})} = \frac{\alpha(\lambda_{m_p})}{\alpha(1+\tilde{F}(\lambda_{m_{p+1}}, 1/u))}.$$

This on proceeding to limits, since $\alpha(x) \in \Omega$, gives that

$$\lambda_\infty(\alpha,\alpha,f) \geq u \liminf_{p\to\infty} \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})}.$$

If $\alpha(x) \in \bar{\Omega}$, we take $r'_p = \exp\left(\frac{u}{u-1}\tilde{F}(\lambda_{m_p}, 1/(u-1))\right)$. Then, for $r'_p \leq r \leq r'_{p+1}$, using Cauchy's inequality and (6.4.1), we have

$$\log M(r,f) \geq \log |a_{m_p}| + \lambda_{m_p} \log r'_p$$
$$> \frac{1}{u-1}\lambda_{m_p}\tilde{F}(\lambda_{m_p}, 1/(u-1))$$

and so

$$\frac{\alpha((u-1)\log M(r,f))}{\alpha(\frac{u-1}{u}\log r)} > \frac{\frac{1}{u-1}\alpha(\lambda_{m_p})}{\alpha(\frac{u-1}{u}\log r'_{p+1})}$$
$$+ \frac{\log \lambda_{m_p}}{\alpha(\frac{u-1}{u}\log r'_{p+1})}\left.\frac{d\alpha(x)}{d\log x}\right|_{x=x^*(\lambda_{m_p})}$$

$$= \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})} + (u-1) \frac{\log \lambda_{m_p}}{\alpha(\lambda_{m_{p+1}})} \left. \frac{d\alpha(x)}{d \log x} \right|_{x=x^*(\lambda_{m_p})}$$

where $\tilde{F}(\lambda_{m_p}, 1/(u-1)) < x^*(\lambda_{m_p}) < \lambda_{m_p} \tilde{F}(\lambda_{m_p}, 1/(u-1))$. Since $\alpha(x) \in \bar{\Omega}$, on passing to limits, the above inequality gives that

$$\lambda_\infty(\alpha,\alpha,f) \geq u \liminf_{p \to \infty} \frac{\alpha(\lambda_{m_p})}{\alpha(\lambda_{m_{p+1}})} .$$

This proves the theorem.

REMARK. Taking $\alpha(x) = \log_p x$, $p \geq 1$, a result of Juneja, Kapoor and Bajpai [45] follows from the above theorem.

CHAPTER 7

APPROXIMATION OF ENTIRE FUNCTIONS AND ENTIRE SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

7.1. The present chapter first deals with the study of the approximation of entire functions over a Caratheodory domain in L^{δ}-norm, $1 \leq \delta < \infty$, and over a compact set having nonzero transfinite diameter in uniform norm. Further, in this chapter, we have studied the approximation of an entire harmonic function in R^{p+2}, $p \geq 1$, on a finite closed hyperball in L^{δ}-norm, $1 \leq \delta \leq \infty$. The approximation of entire generalized axisymmetric potentials (GASP's) and entire generalized biaxisymmetric potentials (GBSP's) on a finite disc in L^{δ}-norm, $1 \leq \delta \leq \infty$, is also studied in this chapter. The results in these directions are obtained by the extensive use of the results of Chapter 6.

In Sections 7.5, 7.6 and 7.7, respectively, the concepts of generalized (α,β)-order and generalized lower (α,β)-order for an entire harmonic function in R^{p+2}, $p \geq 1$, an entire GASP and entire GBSP are introduced. Analogous concepts for entire functions are given by (1.3.3) and Definition 6.1.1.

Using the results of Šeremeta [93] and Shah [102], i.e., (1.3.7), (1.3.10), (1.3.11) and (1.3.12), and the results of Chapter 6, we obtain necessary and sufficient conditions in terms of various approximation errors, such that a function defined on the underlying finite set over which the approximation

is considered, has an analytic continuation as an entire function an entire harmonic function, an entire GASP or an entire GBSP having finite generalized (α,β)-order and finite generalized lower (α,β)-order. The results of this chapter generalize and extend some results of Giroux [35], Kapoor and Nautiyal [57] and McCoy [67, 68, 70].

7.2. Let E denote a Caratheodory domain, i.e., a bounded simply connected domain such that the boundary of E coincides with the boundary of the domain lying in the complement of the closure of E and containing the point ∞. Let $L^\delta(E)$, $1 \leq \delta < \infty$, be the class of all functions $f(z)$ analytic in E and satisfying

$$(7.2.1) \qquad \|f\|_{E,\delta} = \left(\iint_E |f(z)|^\delta \, dx\, dy\right)^{1/\delta} < \infty.$$

Then $\|\cdot\|_{E,\delta}$ is a norm and called the L^δ-norm on $L^\delta(E)$. It is known [107 , p. 393] that $L^2(E)$ forms a Hilbert space with respect to the inner product

$$(f,g) = \iint_E f(z)\,\overline{g(z)}\, dx\, dy, \quad f(z),\ g(z) \in L^2(E).$$

The Fourier coefficients d_n of $f(z) \in L^2(E)$ are defined as

$$(7.2.2) \qquad d_n \equiv d_n(f,E) = \iint_E f(z)\,\overline{p_n(z)}\, dx\, dy, \; n = 0,1,2,\ldots,$$

where $p_n(z)$ is a polynomial of degree n such that $\{p_n(z)\}_{n=0}^{\infty}$ forms a complete orthonormal sequence in $L^2(E)$. Such a sequence always exists in $L^2(E)$ (see, e.g., [66, p. 125], [108, p. 261]). It is known [108, p. 273] that $f(z) \in L^2(E)$ is entire, if and only if,

$$(7.2.3) \qquad \lim_{n \to \infty} |d_n|^{1/n} = 0.$$

Moreover, in that case, $f(z) = \sum_{n=0}^{\infty} d_n p_n(z)$ holds in the whole complex plane.

For $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, we define $\Delta_{n,\delta}(f)$, the error in approximating the function $f(z)$ by polynomials of degree atmost n in L^{δ}-norm as

$$(7.2.4) \quad \Delta_{n,\delta}(f) \equiv \Delta_{n,\delta}(f,E) = \inf_{g \in \mathbb{P}_n} ||f-g||_{E,\delta}, \; n = 0,1,2,\ldots,$$

where $\mathbb{P}_n$ consists of all polynomials of degree atmost n.

In this section we find some preliminary lemmas that are needed in Section 7.3 for approximation of entire functions over a Caratheodory domain in L^{δ}-norm, $1 \leq \delta < \infty$.

To avoid some trivial cases we shall assume throughout in Sections 7.2 and 7.3 that $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, is not a polynomial.

Let E^* be the component of the complement of $\bar{E}$, the closure of the Caratheodory domain E, that contains the point ∞. Let the function $w = \phi(z)$ map E^* conformally onto $|w| > 1$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$ and set $E_r = \{z : |\phi(z)| = r\}$, $r > 1$. For a function $f(z)$, analytic inside and on E_r, let

$$\bar{M}(r,f) = \max_{z \in E_r} |f(z)| \text{ and } M(r,f) = \max_{|z|=r} |f(z)|.$$

We have

LEMMA 7.2.1. *Let $f(z)$ be an entire function. Then*

(i) *For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L_*^0$,*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\beta(\log r)} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \bar{M}(r,f))}{\beta(\log r)}.$$

(ii) *If $f(z)$ is of order $\rho_\infty(f)$ $(0 < \rho_\infty(f) < \infty)$, type $T_\infty(f)$ and lower type $t_\infty(f)$, then*

$$\begin{matrix} T_\infty(f)\, d^{\rho_\infty(f)} \\ t_\infty(f)\, d^{\rho_\infty(f)} \end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \bar{M}(r,f)}{r^{\rho_\infty(f)}},$$

where d is the transfinite diameter of $\bar{E}$, the closure of E.

PROOF. Let z_0 be a point of E and let $|E| = \sup\{|z-\xi| : z, \xi \in \bar{E}\}$ be the diameter of $\bar{E}$. Then, for $z \in E_r$, $r > 1$, following the proof of Lemma 3.1 of Winiarski [132], we have

$$dr - 2|E| - |z_0| < |z| < dr + |E| + |z_0|.$$

The lemma now follows easily from the above inequalities.

LEMMA 7.2.2. *Let $f(z) \in L^2(E)$ has an analytic continuation as an entire function. Then, the function $f^*(z) = \sum_{n=0}^{\infty} |d_n| z^n$, d_n's being the Fourier coefficients of $f(z)$ given by (7.2.2), is entire. Further*

(i) *For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L_*^0$,*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\beta(\log r)} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f^*))}{\beta(\log r)}$$

(ii) *If $f(z)$ is of order $\rho_\infty(f)$ $(0 < \rho_\infty(f) < \infty)$, type $T_\infty(f)$ and lower type $t_\infty(f)$, then $f^*(z)$ is of order $\rho_\infty(f)$, type $T_\infty(f)\, d^{\rho_\infty(f)}$ and lower type $t_\infty(f)\, d^{\rho_\infty(f)}$, where d is the transfinite diameter of $\bar{E}$.*

PROOF. (i) Since $f(z)$ is entire, it follows, from (7.2.3), tha $f^*(z)$ is also an entire function.

If $\{p_n(z)\}_{n=0}^{\infty}$, $p_n(z)$ being a polynomial of degree n, is a complete orthonormal sequence in $L^2(E)$ and $r_* > 1$, we have [108,p. 272].

$$(7.2.5) \qquad |p_n(z)| \le K\, r_*^n \quad \text{for } z \in \bar{E},\ n = 0,1,2,\ldots,$$

where K is a constant and $\bar{E}$ is the closure of E. Now, applying Bernstein inequality ([66,p.112], [108, p. 20], [129,p.77]) to each term of the series $\sum_{n=0}^{\infty} d_n p_n(z)$, we get

$$|f(z)| \le \sum_{n=0}^{\infty} |d_n|\ |p_n(z)| \le K \sum_{n=0}^{\infty} |d_n|\ (rr_*)^n \quad \text{for } z \in E_r,\ r > 1,$$

and so

$$(7.2.6) \qquad \bar{M}(r,f) \le K\, M(rr_*, f^*).$$

The above inequality, for $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L^0_*$, gives that

$$(7.2.7) \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \bar{M}(r,f))}{\beta(\log r)} \le \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f^*))}{\beta(\log r)}\ .$$

Now, let $r' > 1$ be a fixed constant. Since $f(z)$ is entire, it follows that [66,p.114], there exists a sequence of polynomials $\{Q_n(z)\}$, $Q_n(z)$ being of degree of atmost n, such that

$$(7.2.8) \quad |f(z)-Q_n(z)| < \frac{3}{2}\,\bar{M}(r,f)\,\frac{(r'/r)^{n+1}}{1-(r'/r)}, \quad z \in \bar{E},$$

for all sufficiently large values of n and all $r > r'$.

By (7.2.2), we have

$$d_n = \iint_E (f(z)-Q_{n-1}(z))\,\overline{p_n(z)}\,dx\,dy,$$

since $p_n(z)$ is orthogonal to any polynomial of degree less than n. Applying Schwartz's inequality to the above relation, we get

$$|d_n| \leq \left(\iint_E |f(z)-Q_{n-1}(z)|^2\,dx\,dy\right)^{1/2}$$

and so, using (7.2.8), we have

$$(7.2.9) \qquad |d_n| \leq K_o\,\bar{M}(r,f)\,(r'/r)^n$$

for all sufficiently large values of n and all $r > 2r'$(say), where K_o is a constant independent of n and r. Moreover, (7.2.9) gives that

$$(7.2.10) \qquad \mu(r/r', f^*) \leq K_o\,\bar{M}(r,f)$$

for all sufficiently large values of r. For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L^o_*$, the relation (7.2.10), in view of (1.4.8), gives that

$$(7.2.11) \quad \lim_{r\to\infty}\ {\sup \atop \inf}\ \frac{\alpha(\log M(r,f^*))}{\beta(\log r)} \leq \lim_{r\to\infty}\ {\sup \atop \inf}\ \frac{\alpha(\log \bar{M}(r,f))}{\beta(\log r)}.$$

Combining (7.2.7) and (7.2.11) and applying Lemma 7.2.1 to the function f(z), part (i) of the lemma follows.

that $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, has an analytic continuation as an entire function having finite generalized (α,β)-order $\rho_{\infty}(\alpha,$ and finite generalized lower (α,β)-order $\lambda_{\infty}(\alpha,\beta,f)$, given by (1.3.3) and Definition 6.1.1. Finally, we obtain a necessary condition for an entire function to be of generalized regular (α,β)-growth. Our results in this section generalize and exten the results of Giroux [35] and Reddy [85].

First, we obtain a lemma that connects the approximation error $\Delta_{n,\delta}(f)$ of an entire function $f(z)$ with the maximum modul of $f(z)$ on the curve E_r, $r > 1$.

LEMMA 7.3.1. *Let $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, has an analytic continuation as an entire function and let $r' > 1$ and $r_* > 1$ be given. Then, for all sufficiently large values of k and all $r > 2r'r_*$,*

$$\Delta_{k,\delta}(f) \leq K^{o}\, \bar{M}(r,f)\, (r'r_*/r)^{k},$$

where the approximation error $\Delta_{k,\delta}(f)$ is given by (7.2.4) and K^{o} is a constant independent of n and r.

PROOF. Let the Fourier series expansion of the entire function $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, be given as $f(z) = \sum_{n=0}^{\infty} d_n p_n(z)$. Set $g_k(z) = \sum_{n=0}^{k} d_n p_n(z)$. Clearly $g_k \in \mathbb{P}_k$. From the definition (7.2.4) of the approximation error $\Delta_{k,\delta}(f)$, we obtain

$$\Delta_{k,\delta}(f) \leq ||f-g_k||_{E,\delta}$$

and so, by the definition (7.2.1) of the norm $||\cdot||_{E,\delta}$, we have

$$\Delta_{k,\delta}(f) \leq K_* \max_{z \in \bar{E}} |f(z)-g_k(z)|$$

$$\leq K_* \max_{z \in \bar{E}} \left(\sum_{n=k+1}^{\infty} |d_n| \, |p_n(z)| \right),$$

where K_* is a constant. Using (7.2.5) and (7.2.9), for all sufficiently large values of k and $r > 2r'$, we now have

$$\Delta_{k,\delta}(f) \leq K_* K_o K \bar{M}(r,f) \sum_{n=k+1}^{\infty} (r' r_*/r)^n$$

$$= K_* K_o K \bar{M}(r,f) \frac{(r' r_*/r)^{k+1}}{1-(r' r_*)/r} .$$

Taking $r > 2r' r_*$, the lemma follows from the above inequality.

We now prove

<u>THEOREM</u> 7.3.1. <u>Let</u> $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$. <u>Then</u>, $f(z)$ <u>has an analytic continuation as an entire function, if and only if,</u>

$$(7.3.1) \qquad \lim_{n \to \infty} (\Delta_{n,\delta}(f))^{1/n} = 0$$

<u>where the approximation error</u> $\Delta_{n,\delta}(f)$ <u>is given by</u> (7.2.4).

<u>PROOF</u>. First, let $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, be an entire function. Then, it follows, from Lemma 7.3.1, that $\limsup_{n \to \infty} (\Delta_{n,\delta}(f))^{1/n} \leq (r' r_*)/r$ for every $r > 2r' r_*$. Letting $r \to \infty$, this gives

$$\lim_{n \to \infty} (\Delta_{n,\delta}(f))^{1/n} = 0.$$

This proves the necessity part of the theorem.

Now, let $z_o \in E$ and let $R > 0$ be such that $D_{R,z_o} = \{z: |z-z_o| \leq R\}$, is contained in E. If $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$,

then $f(z)$ is holomorphic on D_{R,z_0} and has the following Taylor series expansion

$$(7.3.2) \qquad f(z) = \sum_{n=0}^{\infty} \bar{a}_n (z-z_0)^n$$

on D_{R,z_0}, where $\bar{a}_n$'s are given by

$$\frac{\pi R^{2(n+1)}}{n+1} \bar{a}_n = \iint_{D_{R,z_0}} f(z) \overline{(z-z_0)^n} \, dx \, dy.$$

Thus, for any $g \in \mathbb{P}_{n-1}$, we have

$$\frac{\pi R^{2(n+1)}}{n+1} |\bar{a}_n| = \left| \iint_{D_{R,z_0}} (f(z)-g(z)) \overline{(z-z_0)^n} \, dx \, dy \right|$$

$$\leq R^n \, ||f-g||_{E,1}.$$

Using Holder's inequality, the above relation gives that

$$\frac{\pi R^{n+2}}{n+1} |\bar{a}_n| \leq (A(E))^q \, ||f-g||_{E,\delta}, \; 1 \leq \delta < \infty,$$

where $A(E)$ is the area of E and $q = 1-1/\delta$. Since the above relation holds for any $g \in \mathbb{P}_{n-1}$, we get

$$(7.3.3) \qquad \frac{\pi R^{n+2}}{n+1} |\bar{a}_n| \leq (A(E))^q \, \Delta_{n-1,\delta}(f).$$

If, for $f(z) \in L^\delta(E)$, the equation (7.3.1) holds, then it follows from (7.3.3) that

$$\lim_{n \to \infty} |\bar{a}_n|^{1/n} = 0$$

and so, (7.3.2) gives that $f(z)$ is an entire function. This proves the sufficiency part of the theorem.

The theorem is thus proved.

REMARKS. (i) Let E be a domain bounded by a closed Jordan curve. Then, for $\delta = 2$ a result of Rizvi [87] and for $2 \le \delta < \infty$ a result of Giroux [35] follows from the above theorem.

(ii) For $E = \{z: |z| < 1\}$ and $1 \le \delta < \infty$, some results of Reddy [85] and Ibragimov and Šihaliev [39] follow from this theorem.

(iii) Let E be a Jordan domain whose boundary consists of finite number of analytic Jordan arcs meeting in corners of exterior openings less than or equal to $s\pi < 2\pi$. Then, for $1 \le \delta < \infty$, a result of Rizvi [87,p. 68] follows from Theorem 7.3.1.

Our next lemma is a key result in the proofs of Theorems 7.3.2 to 7.3.6.

LEMMA 7.3.2. *Let* $f(z) \in L^{\delta}(E)$, $1 \le \delta < \infty$, *has an analytic continuation as an entire function. Then the function* $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^n$, *where the approximation errors* $\Delta_{n,\delta}(f)$ *are given by* (7.2.4), *is entire. Further,*

(i) *For* $\alpha(x) \in \Lambda_*$ *and* $\beta(x) \in L^0_*$,

$$\lim_{r\to\infty} {\sup \atop \inf} \frac{\alpha(\log M(r,f))}{\beta(\log r)} = \lim_{r\to\infty} {\sup \atop \inf} \frac{\alpha(\log M(r,f_\delta))}{\beta(\log r)} .$$

(ii) *If* $f(z)$ *is of order* $\rho_\infty(f)$, $0 < \rho_\infty(f) < \infty$, *type* $T_\infty(f)$ *and lower type* $t_\infty(f)$, *then* $f_\delta(z)$ *is of order* $\rho_\infty(f)$, *type*

$T_\infty(f)\, d^{\rho_\infty(f)}$ <u>and lower type</u> $t_\infty(f)\, d^{\rho_\infty(f)}$, <u>where</u> d <u>is the transfinite diameter of</u> $\bar{E}$.

<u>PROOF</u>. (i) Since $f(z)$ is entire, it follows from Theorem 7.3.1 that $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^n$ is also entire. Further, by Lemma 7.3.1, we have

$$(7.3.4) \qquad \mu(r/(r' r_*),\ f_\delta) \leq K^0\, \bar{M}(r,f)$$

for all sufficiently large values of r. For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L_*^0$, the relation (7.3.4), on using (1.4.8), gives that

$$(7.3.5) \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f_\delta))}{\beta(\log r)} \leq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \bar{M}(r,f))}{\beta(\log r)} .$$

Now, by (7.2.2), for any $g \in \mathbb{P}_{n-1}$, we have

$$d_n = \iint_E (f(z)-g(z))\, \overline{p_n(z)}\, dx\, dy,$$

since $p_n(z)$ is orthogonal to any polynomial of degree less than n. In view of the bound (7.2.5) of the polynomials $p_n(z)$, the above relation gives that

$$|d_n| \leq \iint_E |f(z)-g(z)|\ |p_n(z)|\, dx\, dy \leq K\, r_*^n \iint_E |f(z)-g(z)|\, dx\, dy.$$

$$= K\, r_*^n\, ||f-g||_{E,1},$$

where $r_* > 1$. On applying Holder's inequality, for any $g \in \mathbb{P}_{n-1}$, this gives

$$(7.3.6) \qquad |d_n|/r_*^n \leq K(A(E))^q\, ||f-g||_{E,\delta}\ ,\ 1 \leq \delta < \infty,$$

where $A(E)$ is the area of E and $q = 1-1/\delta$. Since (7.3.6) holds for any $g \in \mathbb{P}_{n-1}$, we now have

$$|d_n|/r_*^n \leq K(A(E))^q \, \Delta_{n-1,\delta}(f).$$

This easily gives that

$$\mu(r/r_*, f^*) \leq K(A(E))^q \, r \, \mu(r, f_\delta) \tag{7.3.7}$$

for all sufficiently large values of r, where $f^*(z) = \sum_{n=0}^{\infty} |d_n| z^n$ is an entire function.

Using (7.3.7) and applying (1.4.8) to the functions $f^*(z)$ and $f_\delta(z)$, we get

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f^*))}{\beta(\log r)} \leq \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f_\delta))}{\beta(\log r)} . \tag{7.3.8}$$

Part (i) of the lemma now follows from (7.3.5), (7.3.8), Lemma 7.2.1 and Lemma 7.2.2.

(ii) Part (i) of the lemma gives that if $f(z)$ is of order $\rho_\infty(f)$ then $f_\delta(z)$, $1 \leq \delta < \infty$, is also of order $\rho_\infty(f)$. Now, if $0 < \rho_\infty(f) < \infty$, then let $f(z)$ be of type $T_\infty(f)$ and lower type $t_\infty(f)$. The relation (7.3.4), in view of (1.4.7) and Lemma 7.2.1, now gives that

$$T_\infty(f_\delta) \leq T_\infty(f) \, d^{\rho_\infty(f)}, \quad t_\infty(f_\delta) \leq t_\infty(f) \, d^{\rho_\infty(f)}, \tag{7.3.9}$$

since $r' > 1$ and $r_* > 1$ are arbitrary. Here $T_\infty(f_\delta)$ and $t_\infty(f_\delta)$ are, respectively, type and lower type of $f_\delta(z)$. Now, from (7.3.7), on using (1.4.7) and Lemma 7.2.2, we have

$$(7.3.10) \qquad T_\infty(f)\, d^{\rho_\infty(f)} \le T_\infty(f_\delta), \quad t_\infty(f)\, d^{\rho_\infty(f)} \le t_\infty(f_\delta),$$

since $r_* > 1$ is arbitrary. Part (ii) of the lemma now follows on combining (7.3.9) with (7.3.10).

This proves the lemma.

REMARK. For $E = \{z : |z| < 1\}$ and $\delta = 2$, the lemma is due to Bajpai and Shah [6].

We now have

THEOREM 7.3.2. *Let* $f(z) \in L^\delta(E)$, $1 \le \delta < \infty$, *and let the approximation errors* $\Delta_{n,\delta}(f)$ *be given by* (7.2.4). *A necessary and sufficient condition that* $f(z)$ *has an analytic continuation as an entire function of finite generalized* (α,β)-*order* $\rho_\infty(\alpha,\beta,f)$ *is that*

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.6), *then*

$$(7.3.11) \qquad \rho_\infty(\alpha,\beta,f) = U_\delta(\alpha,\beta).$$

(ii) *If* $\alpha \equiv \beta$ *and* $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$, *then*

$$(7.3.12) \qquad \rho_\infty(\alpha,\alpha,f) = Q(U_\delta(\alpha,\alpha)),$$

where,

$$(7.3.13) \qquad U_\delta(\alpha,\beta) = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta\left(\frac{1}{n}\log\,(\Delta_{n,\delta}(f))^{-1}\right)} < \infty.$$

PROOF. First, let $f(z) \in L^\delta(E)$, $1 \le \delta < \infty$, be an entire function of finite generalized (α,β)-order $\rho_\infty(\alpha,\beta,f)$. Then, by Lemma 7.3.2, the relation (7.3.11) follows on applying (1.3.7)

to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^n$, while the relation (7.3.12) follows on applying Theorem 6.2.1 to the function $f_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, the quantity $U_\delta(\alpha,\beta)$, given by (7.3.13), is finite, then $\lim_{n \to \infty}(\Delta_{n,\delta}(f))^{1/n} = 0$ and so $f(z)$ is entire by Theorem 7.3.1. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

<u>THEOREM</u> 7.3.3. <u>Let</u> $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, <u>and let</u> $\Delta_{n,\delta}(f)/\Delta_{n+1,\delta}(f)$, <u>where the approximation errors</u> $\Delta_{n,\delta}(f)$ <u>are given by</u> (7.2.4), <u>be ultimately a nondecreasing function of</u> n. <u>Then a necessary and sufficient condition that</u> $f(z)$ <u>has an analytic continuation as an entire function of finite generalized lower</u> (α,β)-<u>order</u> $\lambda_\infty(\alpha,\beta,f)$ <u>is that</u>

(i) <u>If</u> $\alpha(x)$ <u>and</u> $\beta(x)$ <u>satisfy</u> (1.3.8) <u>and</u> (1.3.9),

$$\lambda_\infty(\alpha,\beta,f) = V_\delta(\alpha,\beta) \tag{7.3.14}$$

(ii) <u>If</u> $\alpha \equiv \beta$ <u>and</u> $\alpha(x)$ <u>belongs to</u> Ω <u>or</u> $\bar{\Omega}$,

$$\lambda_\infty(\alpha,\alpha,f) = Q(V_\delta(\alpha,\alpha)), \tag{7.3.15}$$

<u>where</u>,

$$V_\delta(\alpha,\beta) = \liminf_{n \to \infty} \frac{\alpha(n)}{\beta(\frac{1}{n} \log (\Delta_{n,\delta}(f))^{-1})} < \infty. \tag{7.3.16}$$

<u>PROOF</u>. First, let $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, be an entire function of finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$ and let

$\Delta_{n,\delta}(f)/\Delta_{n+1,\delta}(f)$ be ultimately a nondecreasing function of n. Then, using Lemma 7.3.2, the relation (7.3.14) follows on applying (1.3.10) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^n$, while (7.3.15) follows on applying Theorem 6.3.1 to the function $f_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if for $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, $\Delta_{n,\delta}(f)/\Delta_{n+1,\delta}(f)$ is ultimately nondecreasing and (7.3.16) holds, then $\lim_{n\to\infty}(\Delta_{n,\delta}(f))^{1/n} = 0$ and so f(z) is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

THEOREM 7.3.4. <u>Let</u> $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, <u>and let the approximation errors</u> $\Delta_{n,\delta}(f)$ <u>be given by</u> (7.2.4). <u>A necessary and sufficient condition that</u> f(z) <u>has an analytic continuation as an entire function having finite generalized lower</u> (α,β)-<u>order</u> $\lambda_\infty(\alpha,\beta,f)$ <u>is that</u>

(i) <u>If</u> $\alpha(x)$ <u>and</u> $\beta(x)$ <u>satisfy</u> (1.3.8) <u>and</u> (1.3.9), <u>then</u>

$$\lambda_\infty(\alpha,\beta,f) = \max_{\{n_k\}} [v_\delta^*(\alpha,\beta,\{n_k\})] = \max_{\{n_k\}} [v_\delta^o(\alpha,\beta,\{n_k\})]. \tag{7.3.17}$$

(ii) <u>If</u> $\alpha \equiv \beta$ <u>and</u> $\alpha(x) \in \Omega$, <u>then</u>

$$\lambda_\infty(\alpha,\alpha,f) = \max_{\{n_k\}} [Q(v_\delta^*(\alpha,\alpha,\{n_k\}))] \tag{7.3.18}$$

$$= \max_{\{n_k\}} [Q(v_\delta^o(\alpha,\alpha,\{n_k\}))].$$

(iii) *If* $\alpha \equiv \beta$, $\alpha(x) \in \bar{\Omega}$ *and the principal indices* $\{n_m\}$ *of the function* $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\ z^n$ *satisfy* $\alpha(n_m) \sim \alpha(n_{m-1})$ *as* $m \to \infty$, *then*

$$(7.3.19)\quad \lambda_\infty(\alpha,\alpha,f) = \max_{\{n_k\}} \left[Q_{\tilde{\chi}(\{n_k\})}(V_\delta^*(\alpha,\alpha,\{n_k\}))\right]$$

$$= \max_{\{n_k\}} \left[Q_{\tilde{\chi}(\{n_k\})}(V_\delta^{o}(\alpha,\alpha,\{n_k\}))\right],$$

where,

$$V_\delta^*(\alpha,\beta,\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k} \log (\Delta_{n_k,\delta}(f))^{-1})},$$

$$V_\delta^{o}(\alpha,\beta,\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k - n_{k-1}} \log (\Delta_{n_{k-1},\delta}(f)/\Delta_{n_k,\delta}(f)))},$$

$$(7.3.20)\qquad \tilde{\chi}(\{n_k\}) = \liminf_{k \to \infty} \alpha(n_{k-1})/\alpha(n_k),$$

maximum in (7.3.17), (7.3.18) *and* (7.3.19) *is taken over all increasing sequences* $\{n_k\}$ *of positive integers and* $\max_{\{n_k\}} [V_\delta^*(\alpha,\beta,\{n_k\})]$ *and* $\max_{\{n_k\}} [V_\delta^{o}(\alpha,\beta,\{n_k\})]$ *are finite.*

PROOF. First, let $f(z) \in L^\delta(E)$, $1 \le \delta < \infty$, be an entire function with finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$. Then, using Lemma 7.3.2, the relation (7.3.17) follows on applying (1.3.11) and (1.3.12) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\ z^n$, while (7.3.18) and (7.3.19) follow on applying Theorem 6.3.2 to the function $f_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, $\max_{\{n_k\}} [V_{\delta}^{*}(\alpha,\beta,\{n_k\})]$ is finite then $\lim_{n \to \infty}(\Delta_{n,\delta}(f))^{1/n} = 0$ and so $f(z)$ is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

We note that the choice $\alpha(x) = x$ and $\beta(x) = (\exp x)^{c}$, $0 < c < \infty$, is not permissible in Theorems 7.3.2 to 7.3.4. Thus, we now obtain characterizations of type and lower type of an entire function in terms of the approximation errors $\Delta_{n,\delta}(f)$, $1 \leq \delta < \infty$, given by (7.2.4).

THEOREM 7.3.5. *Let $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, and let the approximation errors $\Delta_{n,\delta}(f)$ be given by (7.2.4). A necessary and sufficient condition that $f(z)$ has an analytic continuation as an entire function of order $\rho_{\infty}(f)$, $0 < \rho_{\infty}(f) < \infty$, and nonzero finite type $T_{\infty}(f)$ is that*

$$W_{\delta} = \limsup_{n \to \infty} n(\Delta_{n,\delta}(f))^{\rho_{\infty}(f)/n}, \quad 1 \leq \delta < \infty,$$

satisfies $0 < W_{\delta} < \infty$. Further, $W_{\delta} = e\,\rho_{\infty}(f)\, T_{\infty}(f)\, d^{\rho_{\infty}(f)}$, $1 \leq \delta < \infty$, also holds, where d is the transfinite diameter of the closure of E.

PROOF. Necessity part of the theorem follows from Lemma 7.3.2 on applying (1.2.8) to the function $f_{\delta}(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^{n}$. Conversely if, for $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, W_{δ} satisfies $0 < W_{\delta} < \infty$, then we have

$$\limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_{n,\delta}(f)} = \rho_\infty(f)$$

and so, by Theorem 7.3.2, f(z) is an entire function of order $\rho_\infty(f)$. Sufficiency part now follows from the necessity part. This proves the theorem.

THEOREM 7.3.6. *Suppose* $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, *has an analytic continuation as an entire function having order* $\rho_\infty(f), 0 < \rho_\infty(f) < \infty$, *and lower type* $t_\infty(f)$. *If* $\Delta_{n,\delta}(f)/\Delta_{n+1,\delta}(f)$, *where the approxomation error* $\Delta_{n,\delta}(f)$ *is given by* (7.2.4), *is ultimately a nondecreasing function of* n, *then*

$$e\,\rho_\infty(f)\; t_\infty(f)\; d^{\rho_\infty(f)} = \liminf_{n \to \infty} n(\Delta_{n,\delta}(f))^{\rho_\infty(f)/n}.$$

Here d *is the transfinite diameter of the closure of* E.

PROOF. The theorem follows from Lemma 7.3.2 on applying (1.2.15) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^n$.

REMARKS. (i) Taking, in particular $E = \{z : |z| < 1\}$, $\alpha(x) = \log x$ and $\beta(x) = x$, some results of Reddy [85] and Ibragimov and Šihaliev [39] follow from Theorems 7.3.2 and 7.3.5.

(ii) Taking $E = \{z : |z| < 1\}$ and $\delta = 2$ some results of Bajpai and Shah [6] follow from Theorems 7.3.2, 7.3.3 and 7.3.4.

(iii) Let E be a domain bounded by a closed Jordan curve, $\alpha(x) = \log x$, $\beta(x) = x$. For $\delta = 2$, some results of Rizvi [87] follow from Theorems 7.3.2 to 7.3.6.

(iv) Let E be a domain bounded by a closed Jordan curve. Then Theorem 7.3.2, with $\alpha(x) = \log x$ and $\beta(x) = x$, and Theorem 7.3.5 extend the results of Giroux [35], obtained for the case $2 \leq p < \infty$, for $1 \leq p < 2$ as well. Further, for $\alpha(x) = \log x$ and $\beta(x) = x$, Theorem 7.3.2 generalizes a result of Giroux for the case $2 \leq p < \infty$.

(v) Taking E to be a Jordan domain bounded by finite number of analytic Jordan arcs meeting in corners of exterior openings less than or equal to $s\pi < 2\pi$, $\alpha(x) = \log_p x$ and $\beta(x) = \log_q x$, $p \geq q \geq 1$ or $\alpha(x) = \log_p x$, $p \geq 1$, and $\beta(x) = x$, some results of Rizvi [87] follows from Theorems 7.3.2 to 7.3.6.

For $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, let $\{n_j(\delta)\}_{j=0}^{\infty}$, with $n_0(\delta) = 0$ be the sequence of positive integers defined as follows :

$$(7.3.21) \qquad \Delta_{n_{j-1}(\delta),\delta}(f) > \Delta_{n_j(\delta),\delta}(f), \; \Delta_{n,\delta}(f) = \Delta_{n_{j-1}(\delta),\delta}(f)$$

$$\text{for } n_{j-1}(\delta) \leq n < n_j(\delta), j=1,2,3,\ldots$$

We now obtain a relation that shows the influence of this sequence on the growth of an entire function.

<u>THEOREM</u> 7.3.7. <u>Suppose</u> $f(z) \in L^{\delta}(E)$, $1 \leq \delta < \infty$, <u>has an analytic continuation as an entire function having generalized</u> (α,β)-<u>order</u> $\rho_\infty(\alpha,\beta,f)$ <u>and generalized lower</u> (α,β)-<u>order</u> $\lambda_\infty(\alpha,\beta,f)$. <u>Then</u>,

(i) <u>If</u> $\alpha(x)$ <u>and</u> $\beta(x)$ <u>satisfy</u> (1.3.8) <u>and</u> (1.3.9),

$$\lambda_\infty(\alpha,\beta,f) \le \rho_\infty(\alpha,\beta,f) \liminf_{j\to\infty} \alpha(n_j(\delta))/\alpha(n_{j+1}(\delta)).$$

(ii) <u>If</u> $\alpha \equiv \beta$, $\alpha(x) \in \Omega$ <u>and</u> $\lambda_\infty(\alpha,\alpha,f) > 1$,

$$\lambda_\infty(\alpha,\alpha,f) \le \rho_\infty(\alpha,\alpha,f) \liminf_{j\to\infty} \alpha(n_j(\delta))/\alpha(n_{j+1}(\delta)).$$

(iii) <u>If</u> $\alpha \equiv \beta$, $\alpha(x) \in \bar{\Omega}$,

$$\lambda_\infty(\alpha,\alpha,f)-1 \le (\rho_\infty(\alpha,\alpha,f)-1) \liminf_{j\to\infty} \alpha(n_j(\delta))/\alpha(n_{j+1}(\delta)$$

<u>where the sequence</u> $\{n_j(\delta)\}$ <u>is given by</u> (7.3.21).

<u>PROOF</u>. Since $f(z)$ is entire, it follows, from Lemma 7.3.2, th the entire function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(f)\, z^n$, $1 \le \delta < \infty$, where th approximation error $\Delta_{n,\delta}(f)$ is given by (7.2.4), is of generali (α,β)-order $\rho_\infty(\alpha,\beta,f)$ and generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,$ Hence the function $\bar{f}_\delta(z) = z\, f_\delta(z) - f_\delta(z)$ is also an entire function with generalized (α,β)-order $\rho_\infty(\alpha,\beta,f)$ and generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$. Further

$$(7.3.22) \qquad \bar{f}_\delta(z) = -\Delta_{0,\delta}(f) + \sum_{n=1}^{\infty} (\Delta_{n-1,\delta}(f) - \Delta_{n,\delta}(f))\, z^n$$

$$= -\Delta_{0,\delta}(f) + \sum_{j=1}^{\infty} \Delta_j\, z^{n_j(\delta)}$$

where $\Delta_j = \Delta_{n_{j-1}(\delta),\delta}(f) - \Delta_{n_j(\delta),\delta}(f)$.

Now, if $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9), applying (1.3.5) and (1.3.11) to the function $\bar{f}_\delta(z)$, we get

$$\lambda_\infty(\alpha,\beta,f) = \max_{\{j_k\}}\{\liminf_{k\to\infty} \frac{\alpha(n_{j_{k-1}}(\delta))}{\beta(\frac{1}{n_{j_k}(\delta)}\log(\Delta_{j_k})^{-1})}\}$$

$$\leq \max_{\{j_k\}}\{\limsup_{k\to\infty} \frac{\alpha(n_{j_k}(\delta))}{\beta(\frac{1}{n_{j_k}(\delta)}\log(\Delta_{j_k})^{-1})}\}\times$$

$$\times \max_{\{n_k\}}\{\liminf_{k\to\infty}\frac{\alpha(n_{j_{k-1}}(\delta))}{\alpha(n_{j_k}(\delta))}\}$$

$$\leq \rho_\infty(\alpha,\beta,f)\ \liminf_{j\to\infty} \alpha(n_{j-1}(\delta))/\alpha(n_j(\delta)).$$

This proves part (i) of the theorem.

If $\alpha \equiv \beta$ and $\alpha(x)$ belongs to Ω or $\bar{\Omega}$, it is easily seen that $\bar{f}_\delta(z)$ satisfies the hypothesis of Corollary 6.3.1. Now, parts (ii) and (iii) of the theorem follow on applying Corollar 6.3.1 to the function $\bar{f}_\delta(z)$.

This proves the theorem.

COROLLARY. *Suppose* $f(z) \in L^\delta(E)$, $1 \leq \delta < \infty$, *has an analytic continuation as an entire function of generalized regular* (α,β)-*growth with* (α,β)-*order* $\rho_\infty(\alpha,\beta,f)$. *If*

(i) $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.8) *and* (1.3.9) *and*

$$0 < \rho_\infty(\alpha,\beta,f) < \infty,$$

or

(ii) $\alpha \equiv \beta$, $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$ *and* $1 < \rho_\infty(\alpha,\alpha,f) < \infty$

then

$$\alpha(n_j(\delta)) \sim \alpha(n_{j+1}(\delta)) \quad \textit{as } j \to \infty,$$

where the sequence $\{n_j(\delta)\}$ *is given by* (7.3.21).

REMARKS. (I) It is seen from the above corollary that for an entire function $f(z)$ of generalized regular (α,β)-growth the gaps in the sequence $\{n_j(\delta)\}$, defined by (7.3.21), are of the same order for any δ, $1 \leq \delta < \infty$.

(II) If for an entire function $f(z)$, the sequence $\{n_j(\delta)\}$ $1 \leq \delta < \infty$, given by (7.3.21), has wide gaps, i.e., $\liminf_{j \to \infty} \alpha(n_j(\delta))/\alpha(n_{j+1}(\delta)) < 1$, then, by Theorem 7.3.7, $f(z)$ is of generalized irregular (α,β)-growth if

(i) $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9) and $0 < \rho_\infty(\alpha,\beta,f) < \infty$.

or

(ii) $\alpha \equiv \beta$, $\alpha(x)$ belongs to Ω or $\bar{\Omega}$ and $1 < \rho_\infty(\alpha,\alpha,f) < \infty$.

(III) If E is a Jordan domain, $\delta = 2$, $\alpha(x) = \log_p x$ and $\beta(x) = \log_q x$, $p > q \geq 0$, then a result of Rizvi [87, p. 58] follows from the above theorem.

7.4. In this section, using the results of Chapter 6, we study the behaviour of the rate of decay of the approximation error in uniform norm on a compact set of an entire function of slow growth. Thus, we have obtained necessary and sufficient conditions, in terms of the approximation error in uniform norm over a compact set $\mathcal{E}$ with nonzero transfinite diameter, such that a function $f(z)$ continuous on $\mathcal{E}$ has an analytic continuation as an entire function having finite generalized (α,α)-order

and generalized lower (α,α)-order, given by Definition 6.1.1. Our results generalize the results in [57] which were obtained for the case when $\mathcal{E}$ is taken to be the closed interval $[-1,1]$.

Let $\mathcal{E}$ be a compact set in the complex plane such that the transfinite diameter $d(\mathcal{E})$ of $\mathcal{E}$ is nonzero and let $\mathcal{H}(\mathcal{E})$ be the class of all functions $f(z)$ continuous on $\mathcal{E}$. For $f(z) \in \mathcal{H}(\mathcal{E})$, set

$$||f||_{\mathcal{E},\infty} = \sup_{z \in \mathcal{E}} |f(z)| .$$

Then, $||\cdot||_{\mathcal{E},\infty}$ is called the uniform norm on $\mathcal{H}(\mathcal{E})$. For $f(z) \in \mathcal{H}(\mathcal{E})$, the error $\Delta_{n,\infty}(f)$ in approximating the functic $f(z)$ by polynomials of degree atmost n in uniform norm is defined as

$$(7.4.1) \qquad \Delta_{n,\infty}(f) = \inf_{g \in \mathbb{P}_n} ||f-g||_{\mathcal{E},\infty}$$

where $\mathbb{P}_n$ consists of all polynomials of degree atmost n.

It is known [132] that $f(z) \in \mathcal{H}(\mathcal{E})$ has an analytic continuation as an entire function, if and only if,

$$(7.4.2) \qquad \lim_{n \to \infty} (\Delta_{n,\infty}(f))^{1/n} = 0 .$$

If $\eta^{(n)} = \{\eta_{no}, \eta_{n1}, \ldots, \eta_{nn}\}$ is nth extremal system (see Section 1.6) of $\mathcal{E}$, then the polynomials

$$p^{(j)}(z, \eta^{(n)}) = \prod_{\substack{k=0 \\ j \neq k}}^{n} \frac{z - \eta_{nk}}{\eta_{nj} - \eta_{nk}} , \quad j = 0,1,2,\ldots,n,$$

are called the nth Lagrange's extremal polynomials. Further, there exists finite limit ([60])

$$\lim_{n \to \infty} |P^{(O)}(z, \eta^{(n)})|^{1/n} = L(z) \geq 1$$

for every z in the unbounded component $\mathcal{E}_\infty$ of the complement of $\mathcal{E}$ and L(z) is the modulus of an analytic function $\phi(z)$ in $\mathcal{E}_\infty$ which has a univalent branch

$$\phi(z) = \frac{z}{d(\mathcal{E})} + r_o + \frac{r_1}{z} + \cdots$$

in a neighbourhood of infinity. Set

$$\mathcal{E}_r = \{z : L(z) = r\}$$

and

$$\bar{M}(r,f) = \max_{z \in \mathcal{E}_r} |f(z)|.$$

To avoid some trivial cases we shall assume throughout this section that $f(z) \in \mathcal{H}(\mathcal{E})$ is not a polynomial.

We now have

<u>LEMMA</u> 7.4.1. <u>Let</u> f(z) <u>be an entire function</u>. <u>Then</u>, <u>for</u> $\alpha(x) \in \Lambda_*$,

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f))}{\alpha(\log r)} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log \bar{M}(r,f))}{\alpha(\log r)} .$$

<u>PROOF</u>. Let z_o be a point of $\mathcal{E}$ and let $|\mathcal{E}| = \sup\{|z-\xi| : z, \xi \in \mathcal{E}\}$ be the diameter of $\mathcal{E}$. Then, for $z \in \mathcal{E}_r$, $r > 1$, by the proof of Lemma 3.1 of Winiarski [132], we have

$$d(\mathcal{E})r - 2|\mathcal{E}|-|z_o| < |z| < d(\mathcal{E})r + |\mathcal{E}|+|z_o|.$$

The lemma follows easily on using the above inequalities.

The following lemma is needed in the proofs of Theorems 7.4.1 to 7.4.3. The lemma is essentially due to Bajpai and Shah [6], however, for the sake of completeness we give its proofs.

LEMMA 7.4.2. *Suppose* $f(z) \in \mathcal{H}(\mathcal{E})$ *has an analytic continuation as an entire function. Then,* $f_\infty(z) = \sum_{n=0}^{\infty} \Delta_{n,\infty}(f)\, z^n$, *where the approximation error* $\Delta_{n,\infty}(f)$ *is given by* (7.4.1), *is an entire function. Further, for* $\alpha(x) \in \Lambda_*$,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f))}{\alpha(\log r)} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f_\infty))}{\alpha(\log r)}.$$

PROOF. Since $f(z)$ is entire it follows from (7.4.2) that $f_\infty(z) = \sum_{n=0}^{\infty} \Delta_{n,\infty}(f) z^n$ is an entire function. Further, by [132], there exists a sequence of polynomials $\{q_n(z)\}$, $q_n(z)$ being of degree n, such that, given $\varepsilon > 0$, we have

$$\|f-q_n\|_{\mathcal{E},\infty} \le K\,\bar{M}(r,f)\; e^{n\varepsilon}/r^n$$

for all sufficiently large values of r and n, where K is a constant. Thus, using the definition (7.4.1) of the approximation error $\Delta_{n,\infty}(f)$, we have

$$\Delta_{n,\infty}(f) \le K\,\bar{M}(r,f)\; e^{n\varepsilon}/r^n$$

for all sufficiently large values of r and n. This gives that

$$(7.4.3) \qquad \mu(r/e^{\varepsilon}, f_\infty) \le K\,\bar{M}(r,f)$$

for all sufficiently large values of r. For $\alpha(x) \in \Lambda_*$, the relation (7.4.3), on applying (1.4.8) to the function $f_\infty(z)$, gives that

$$(7.4.4) \quad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f_\infty))}{\alpha(\log r)} \le \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \bar{M}(r,f))}{\alpha(\log r)} .$$

It is known that, for $f(z) \in \mathcal{H}(\mathcal{E})$ there exist polynomials $p_n^{(\infty)}(z) \in \mathbb{P}_n$, $n = 0,1,2,\ldots,$ such that

$$\Delta_{n,\infty}(f) = \|f-p_n^{(\infty)}\|_{\mathcal{E},\infty} .$$

Hence

$$\|p_{n+1}^{(\infty)}-p_n^{(\infty)}\|_{\mathcal{E},\infty} \le \|f-p_n^{(\infty)}\|_{\mathcal{E},\infty} + \|f-p_{n+1}^{(\infty)}\|_{\mathcal{E},\infty}$$

$$\le 2\,\Delta_{n,\infty}(f),$$

and so Bernstein Walsh inequality (see Theorem 2.2 and Corollary 9.1 in [104]), gives

$$(7.4.5) \quad |p_{n+1}^{(\infty)}(z)-p_n^{(\infty)}(z)| \le 2\,\Delta_{n,\infty}(f)\, r^{n+1} \text{ for } z \in \mathcal{E}_r.$$

Since f(z) is entire, it follows from (7.4.2) and (7.4.5) that the series $\sum_{n=0}^{\infty} (p_{n+1}^{(\infty)}(z)-p_n^{(\infty)}(z)) + p_o^{(\infty)}(z)$ converges uniformly on compact subsets of the whole plane and so it represents an entire function. But, on $\mathcal{E}$, $\sum_{n=0}^{\infty} (p_{n+1}^{(\infty)}(z)-p_n^{(\infty)}(z)) + p_o^{(\infty)}(z)$ converges uniformly to the function f(z). Thus f(z) = $\sum_{n=0}^{\infty} (p_{n+1}^{(\infty)}(z)-p_n^{(\infty)}(z)) + p_o^{(\infty)}(z)$ holds in the whole complex plane.

Using (7.4.5), we get

$$\bar{M}(r,f) \le K_o + \sum_{n=0}^{\infty} |p_{n+1}^{(\infty)}(z) - p_n^{(\infty)}(z)| \le K_o + 2\sum_{n=0}^{\infty} \Delta_{n,\infty}(f)\ r^{n+1}$$

$$= K_o + 2r\, M(r, f_\infty).$$

This easily gives that

$$(7.4.6) \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \bar{M}(r,f))}{\alpha(\log r)} \le \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f_\infty))}{\alpha(\log r)}.$$

Combining (7.4.4) and (7.4.6) and using Lemma 7.4.1, the lemma follows.

We now have

<u>THEOREM</u> 7.4.1. <u>Let</u> $f(z) \in \mathcal{H}(\mathcal{E})$ <u>and let the approximation error</u> $\Delta_{n,\infty}(f)$ <u>be given by</u> (7.4.1). <u>Then, a necessary and sufficient condition that</u> $f(z)$ <u>has an analytic continuation as an entire function having finite generalized</u> (α,α)-<u>order</u> $\rho_\infty(\alpha,\alpha,f)$, <u>where</u> $\alpha(x)$ <u>belongs to</u> Ω <u>or</u> $\bar{\Omega}$, <u>is that</u>

$$\rho_\infty(\alpha,\alpha,f) = Q(U_\infty)$$

<u>where</u>

$$(7.4.7) \qquad U_\infty = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(\frac{1}{n}\log(\Delta_{n,\infty}(f))^{-1})} < \infty.$$

<u>PROOF</u>. Necessity part of the theorem follows from Lemma 7.4.2, on applying Theorem 6.2.1 to the function $f_\infty(z) = \sum_{n=0}^{\infty} \Delta_{n,\infty}(f) z^n$. Conversely, if for $f(z) \in \mathcal{H}(\mathcal{E})$, the quantity U_∞, as given by (7.4.7), is finite, then $\lim_{n\to\infty} (\Delta_{n,\infty}(f))^{1/n} = 0$ and so, by (7.4.2), $f(z)$ is entire. Sufficiency part of the theorem now follows from the necessity part. This proves the theorem.

THEOREM 7.4.2. *Let* $f(z)\ \varepsilon\ \mathcal{H}(\mathcal{E})$ *and let* $\Delta_{n,\infty}(f)/\Delta_{n+1,\infty}(f)$, *where the approximation error* $\Delta_{n,\infty}(f)$ *is given by* (7.4.1), *be ultimately a nondecreasing function of* n. *Then, a necessary and sufficient condition that* f(z) *has an analytic continuation as an entire function of finite generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$, *where* $\alpha(x)$ *belongs to* Ω *or to* $\bar{\Omega}$, *is that*

$$\lambda_\infty(\alpha,\alpha,f) = Q(V_\infty)$$

where

$$(7.4.8) \qquad V_\infty = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha(\frac{1}{n}\log(\Delta_{n,\infty}(f))^{-1})} < \infty.$$

PROOF. Necessity part of the theorem follows from Lemma 7.4.2 on applying Theorem 6.3.1 to the function $f_\infty(z) = \sum_{n=0}^{\infty} \Delta_{n,\infty}(f) z^n$. Conversely, if for $f(z)\ \varepsilon\ \mathcal{H}(\mathcal{E})$, $\Delta_{n,\infty}(f)/\Delta_{n+1,\infty}(f)$ is ultimately a nondecreasing function of n and the quantity V_∞, as given by (7.4.8), is finite, then $\lim_{n\to\infty}(\Delta_{n,\infty}(f))^{1/n} = 0$ and so f(z) is entire. Sufficiency part of the theorem now follows from the necessity part. This proves the theorem.

THEOREM 7.4.3. *Let* $f(z)\ \varepsilon\ \mathcal{H}(\mathcal{E})$ *and let the approximation error* $\Delta_{n,\infty}(f)$ *be given by* (7.4.1). *A necessary and sufficient condition that* f(z) *has an analytic continuation as an entire function having finite generalized lower* (α,α)-*order* $\lambda_\infty(\alpha,\alpha,f)$ *is that* (i) *If* $\alpha(x)\ \varepsilon\ \Omega$, *then*

$$(7.4.9) \quad \lambda_\infty(\alpha,\alpha,f) = \max_{\{n_k\}} [Q(V_\infty^*(\alpha,\{n_k\}))] = \max_{\{n_k\}} [Q(V_\infty^{o}(\alpha,\{n_k\}))].$$

(ii) *If* $\alpha(x) \in \bar{\Omega}$ *and the principal indices* $\{n_m\}$ *of the function* $f_\infty(z) = \sum_{n=0}^{\infty} \Delta_{n,\infty}(f)\, z^n$ *satisfy* $\alpha(n_m) \sim \alpha(n_{m+1})$ *as* $m \to \infty$, *then*

$$(7.4.10) \quad \lambda_\infty(\alpha,\alpha,f) = \max_{\{n_k\}} \left[Q_{\tilde{\chi}(\{n_k\})} (V_\infty^*(\alpha,\{n_k\})) \right]$$

$$= \max_{\{n_k\}} \left[Q_{\tilde{\chi}(\{n_k\})} (V_\infty^{o}(\alpha,\{n_k\})) \right],$$

where

$$V_\infty^*(\alpha,\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(n_{k-1})}{\alpha(\frac{1}{n_k} \log (\Delta_{n_k,\infty}(f))^{-1})},$$

$$V_\infty^{o}(\alpha,\{n_k\}) = \liminf_{k \to \infty} \frac{\alpha(n_{k-1})}{\alpha(\frac{1}{n_k - n_{k-1}} \log (\Delta_{n_{k-1},\infty}(f)/\Delta_{n_k,\infty}(f)))},$$

$\tilde{\chi}(\{n_k\})$ *is given by* (7.3.20), *maximum in* (7.4.9) *and* (7.4.10) *is taken over all increasing sequences* $\{n_k\}$ *of positive integers and* $\max_{\{n_k\}} [V_\infty^*(\alpha,\{n_k\})]$ *and* $\max_{\{n_k\}} [V_\infty^{o}(\alpha,\{n_k\})]$ *are finite.*

PROOF. The necessity part of the theorem follows from Lemma 7.4.2, on applying Theorem 6.3.2 to the function $f_\infty(z) = \sum_{n=0}^{\infty} \Delta_{n,\infty}(f)\, z^n$. Conversely, *if, for* $f(z) \in \mathcal{H}(\mathcal{E})$, the quantity $\max_{\{n_k\}} [V_\infty^*(\alpha,\{n_k\})]$ is finite, then $\lim_{n \to \infty} (\Delta_{n,\infty}(f))^{1/n} = 0$ and so $f(z)$ is entire. Sufficiency part of the theorem now follows from the necessity part. This proves the theorem.

REMARKS. (i) Taking $\alpha(x) = \log x$ and $\mathcal{E} = [-1,1]$ some results of Reddy ([83], [84]) follow from Theorems 7.4.1 and 7.4.2.

(ii) With $\alpha(x) = \log_p x$, $p \geq 1$ and $\mathcal{E} = [-1,1]$ some results of Kapoor [50] follow from the results of this section.

(iii) Theorems 7.4.1, 7.4.2 and 7.4.3 are due to Kapoo. and Nautiyal [57] for the case $\mathcal{E} = [-1,1]$.

(iv) Taking $\alpha(x) = \log_p x$, $p \geq 1$, some results of Rizvi [87] follow from the results of this section.

(v) Our results supplement the results of Bajpai and Shah [6], who have obtained the characterizations, in terms of the approximation error $\Delta_{n,\infty}(f)$, of the generalized (α,β)-orde $\rho_\infty(\alpha,\beta,f)$ when $\alpha(x)$ and $\beta(x)$ satisfy (1.3.6) and of the generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,f)$ when $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9).

7.5. In R^{p+2}, a harmonic function H, given by (4.1.2), is said to be entire if the series (4.1.2) of H converges uniformly on compact subsets of R^{p+2}.

Let $\bar{H}_{R_o}$, $0 < R_o < \infty$, as in Section 4.4, denote the class of all harmonic functions regular on the closed hyperball $\bar{B}_{R_o}$ of radius R_o and centered at the origin in R^{p+2}. For $H \in \bar{H}_{R_o}$, let the uniform norm $||\cdot||_{R_o,\infty}$ and L^δ-norm $||\cdot||_{R_o,\delta}$, $1 \leq \delta < \infty$, be given by (4.4.1) and (4.4.2), respectively. Further, for $H \in \bar{H}_{R_o}$, let $\Delta_{n,\infty}(H,R_o)$ and $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta < \infty$, the errors in approximating the function H by harmonic polynomials of degree atmost n, respectively, in uniform norm and L^δ-norm be given by (4.4.3) and (4.4.4). We note, from Theorem 4.4.1, that

$H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire harmonic function, if and only if

(7.5.1) $$\lim_{n \to \infty} (\Delta_{n,\delta}(H,R_o))^{1/n} = 0, \; 1 \leq \delta \leq \infty.$$

In this section, we first introduce the concepts of generalized (α,β)-order and generalized lower (α,β)-order for an entire harmonic function. Then, we obtain necessary and sufficient conditions, in terms of the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, such that $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire harmonic function having finite generalized (α,β)-order and finite generalized lower (α,β)-order. Finally, the characterizations of the type and lower type of an entire harmonic function, as given by (1.11.4) and (1.11.5), are obtained in terms of the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \leq \delta \leq \infty$, $0 < R_o < \infty$. Our results in this section generalize some results in [58].

We first have the following definition.

DEFINITION 7.5.1. *An entire harmonic function* H *in* R^{p+2}, $p = 1,2,\ldots$, *is said to be of generalized* (α,β)-*order* $\rho_\infty(\alpha,\beta,H)$ *and generalized lower* (α,β)-*order* $\lambda_\infty(\alpha,\beta,H)$ *if*

$$\begin{matrix}\rho_\infty(\alpha,\beta,H) \\ \lambda_\infty(\alpha,\beta,H)\end{matrix} = \lim_{r \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log M(r,H))}{\beta(\log r)}$$

where $\alpha(x) \in \Lambda_*$, $\beta(x) \in L^o_*$ *and* $M(r,H)$ *is the maximum modulus of* H *over the hypersphere* S_r *of radius* r *centered at the origin*.

Taking $\alpha(x) = \log x$ and $\beta(x) = x$ in Definition 7.5.1, $\rho_\infty(\alpha,\beta,H)$ becomes the order $\rho_\infty(H)$ of an entire harmonic function H.

To avoid some trivial cases we shall assume in this section that $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, is not a harmonic polynomial

The following lemma is a key result in the proofs of Theorems 7.5.1 to 7.5.5.

<u>LEMMA</u> 7.5.1. <u>Let</u> $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, <u>has an analytic continuation as an entire harmonic function and let</u> $1 \le \delta \le \infty$. <u>Then, the function</u> $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)\ (z/R_o)^n$, <u>where the approximation errors</u> $\Delta_{n,\delta}(H,R_o)$, $1 \le \delta \le \infty$, <u>are given by</u> (4.4.3) <u>and</u> (4.4.4), <u>is entire</u>. <u>Further</u>

(i) <u>For</u> $\alpha(x) \in \Lambda_*$ <u>and</u> $\beta(x) \in L^o_*$,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,H))}{\beta(\log r)} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,h_\delta))}{\beta(\log r)} .$$

(ii) <u>If</u> H <u>is of order</u> $\rho_\infty(H)$, $0 < \rho_\infty(H) < \infty$, <u>type</u> $T_\infty(H)$ <u>and lower type</u> $t_\infty(H)$, <u>then</u> $h_\delta(z)$ <u>is also of order</u> $\rho_\infty(H)$, <u>type</u> $T_\infty(H)$ <u>and lower type</u> $t_\infty(H)$.

<u>PROOF</u>. (i) Since H is entire, it follows from (7.5.1) that $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)\ (z/R_o)^n$ is also entire. Now, since

$$\{(4(n+1)+2p)^{p(p+1)/2}\ (n+p+1)^p\ (n+p/2+1)^{p/2}\}^{1/(n+1)} \to 1$$

as $n \to \infty$, given $r' > 1$, we have

$$(4(n+1)+2p)^{p(p+1)/2}\ (n+p+1)^p\ (n+p/2+1)^{p/2} < (r')^{n+1}$$

for all sufficiently large values of n. Thus, using (4.4.21),

we have

$$\Delta_{n,\delta}(H,R_o) \leq K_\delta^* K^* M(r,H) \ (R_o \ r'/r)^{n+1}$$

for all sufficiently large values of r and n, where K_δ^* and K^* are constants. This gives that

$$\mu(r/r', h_\delta) \leq K_\delta^* K^* M(r,H), \tag{7.5.2}$$

for all sufficiently large values of r. For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L_*^o$, the above relation, on using (1.4.8), gives that

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,h_\delta))}{\beta(\log r)} \leq \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,H))}{\beta(\log r)}. \tag{7.5.3}$$

Now, since

$$\{(4n+2p)^{p(p+1)} \ (n+p/2)^p \ (n+p)^p \ (n+p+2)\}^{1/n} \to 1$$

as $n \to \infty$, given $r_* > 1$ there exists a constant K_{r_*} such that, for $n \geq 1$, we have

$$(4n+2p)^{p(p+1)} (n+p/2)^p \ (n+p)^p \ (n+p+2) \leq K_{r_*} \ r_*^n.$$

Thus, from (4.4.33), we get

$$\begin{aligned} M(r,H) &\leq 2 \ K_*(A_o+K_{r_*} \ \bar{K}_\delta \ \sum_{n=1}^{\infty} \Delta_{n-1,\delta}(H,R_o)(rr_*/R_o)^n) \\ &= 2 \ K_*(A_o+K_{r_*} \ \bar{K}_\delta \ (rr_*/R_o) \ M(rr_*,h_\delta)) \end{aligned} \tag{7.5.4}$$

where K_*, $\bar{K}_\delta$ and A_o are constants. This gives that

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,H))}{\beta(\log r)} \leq \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,h_\delta))}{\beta(\log r)}. \tag{7.5.5}$$

Combining (7.5.3) and (7.5.5) we get Part (i) of the lemma.

(ii) From Part (i) of the lemma it follows that if the entire harmonic function H is of order $\rho_\infty(H)$, then $h_\delta(z)$, $1 \le \delta \le \infty$, is an entire function of order $\rho_\infty(H)$. Now, if $0 < \rho_\infty(H) < \infty$, let H be of type $T_\infty(H)$ and lower type $t_\infty(H)$. In view of (1.4.7), from (7.5.2), we now have

(7.5.6) $$T_\infty(h_\delta) \le T_\infty(H), \; t_\infty(h_\delta) \le t_\infty(H),$$

since $r' > 1$ is arbitrary. Here $T_\infty(h_\delta)$ and $t_\infty(h_\delta)$ are, respectively, type and lower type of $h_\delta(z)$. Now, since $r_* > 1$ is arbitrary, from (7.5.4), we have

(7.5.7) $$T_\infty(H) \le T_\infty(h_\delta), \quad t_\infty(H) \le t_\infty(h_\delta).$$

Part (ii) of the lemma now follows from (7.5.6) and (7.5.7).

This completes the proof of the lemma.

We now have

THEOREM 7.5.1. *Let the harmonic function* $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, *and let the approximation errors* $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, *be given by* (4.4.3) *and* (4.4.4). *A necessary and sufficient condition that* H *has an analytic continuation as an entire harmonic function having finite generalized* (α,β)-*order* $\rho_\infty(\alpha,\beta,H)$ *is that*

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.6), *then*

(7.5.8) $$\rho_\infty(\alpha,\beta,H) = U_H^\delta(\alpha,\beta).$$

(ii) *If* $\alpha \equiv \beta$ *and* $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$, *then*

(7.5.9) $$\rho_\infty(\alpha,\alpha,H) = Q(U_H^\delta(\alpha,\alpha)),$$

where,

$$(7.5.10) \qquad U_H^{\delta}(\alpha,\beta) = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(\frac{1}{n} \log (\Delta_{n,\delta}(H,R_o))^{-1})} < \infty.$$

PROOF. First, let $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire harmonic function having finite generalized (α,β)-order $\rho_\infty(\alpha,\beta,H)$. Then, from Lemma 7.5.1, the relation (7.5.8) follows on applying (1.3.7) to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$, $1 \le \delta \le \infty$, while the relation (7.5.9) follows on applying Theorem 6.2.1 to the function $h_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, the quantity $U_H^{\delta}(\alpha,\beta)$, given by (7.5.10), is finite, then $\lim_{n \to \infty}(\Delta_{n,\delta}(H,R_o))^{\frac{1}{n}} = 0$ and so, by (7.5.1), H is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

REMARK. For $\alpha(x) = \log x$, $\beta(x) = x$ and $p = 1$, a result in [58] follows from Theorem 7.5.1.

THEOREM 7.5.2. Let the harmonic function $H \in \bar{H}_{R_o}$, $0 < R_o < \infty$, and let the approximation errors $\Delta_{n,\delta}(H,R_o)$, $1 \le \delta \le \infty$, be given by (4.4.3) and (4.4.4). Assume that $\Delta_{n,\delta}(H,R_o)/\Delta_{n+1,\delta}(H,R_o)$ is ultimately a nondecreasing function of n. A necessary and sufficient condition that H has an analytic continuation as an entire harmonic function having finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,H)$ is that

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.8) *and* (1.3.9), *then*

(7.5.11) $$\lambda_\infty(\alpha,\beta,H) = V_H^\delta(\alpha,\beta)$$

(ii) *If* $\alpha \equiv \beta$ *and* $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$, *then*

(7.5.12) $$\lambda_\infty(\alpha,\alpha,H) = Q(V_H^\delta(\alpha,\alpha)),$$

where

(7.5.13) $$V_H^\delta(\alpha,\beta) = \liminf_{n\to\infty} \frac{\alpha(n)}{\beta(\frac{1}{n}\log(\Delta_{n,\delta}(H,R_0))^{-1})} < \infty.$$

PROOF. First, let $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, be an entire harmonic function of finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,H)$ and let $\Delta_{n,\delta}(H,R_0)/\Delta_{n+1,\delta}(H,R_0)$ be ultimately a nondecreasing function of n. Then, using Lemma 7.5.1, the relation (7.5.11) follows on applying (1.3.10) to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_0)(z/R_0)^n$, while (7.5.12) follows on applying Theorem 6.3.1 to the function $h_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $H \in \bar{H}_{R_0}$, $\Delta_{n,\delta}(H,R_0)/\Delta_{n+1,\delta}(H,R_0)$ is ultimately nondecreasing and (7.5.13) holds, then $\lim_{n\to\infty}(\Delta_{n,\delta}(H,R_0))^{1/n} = 0$ and so H is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

THEOREM 7.5.3. *Let the harmonic function* $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, *and let the approximation errors* $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, *be given by* (4.4.3) *and* (4.4.4). *A necessary and sufficient condition*

that H has an analytic continuation as an entire harmonic function having finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,H)$ is that

(i) If $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9), then

$$(7.5.14)\quad \lambda_\infty(\alpha,\beta,H) = \max_{\{n_k\}} [\tilde{V}_H^\delta(\alpha,\beta,\{n_k\})] = \max_{\{n_k\}} [\bar{V}_H^\delta(\alpha,\beta,\{n_k\})]$$

(ii) If $\alpha \equiv \beta$ and $\alpha(x) \in \Omega$, then

$$(7.5.15)\quad \lambda_\infty(\alpha,\alpha,H) = \max_{\{n_k\}} [Q(\tilde{V}_H^\delta(\alpha,\alpha,\{n_k\}))]$$

$$= \max_{\{n_k\}} [Q(\bar{V}_H^\delta(\alpha,\alpha,\{n_k\}))] .$$

(iii) If $\alpha \equiv \beta$, $\alpha(x) \in \bar{\Omega}$ and the principal indices $\{n_m\}$ of the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$ satisfy $\alpha(n_m) \sim \alpha(n_{m-1})$ as $m \to \infty$, then

$$(7.5.16)\quad \lambda_\infty(\alpha,\alpha,H) = \max_{\{n_k\}} [Q_{\tilde{\chi}(\{n_k\})}(\tilde{V}_H^\delta(\alpha,\alpha,\{n_k\}))]$$

$$= \max_{\{n_k\}} [Q_{\tilde{\chi}(\{n_k\})}(\bar{V}_H^\delta(\alpha,\alpha,\{n_k\}))] ,$$

where

$$\tilde{V}_H^\delta(\alpha,\beta,\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k}\log(\Delta_{n_k,\delta}(H,R_o))^{-1})} ,$$

$$\bar{V}_H^\delta(\alpha,\beta,\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k-n_{k-1}}\log(\Delta_{n_{k-1},\delta}(H,R_o)/\Delta_{n_k,\delta}(H,R_o)))} ,$$

$\tilde{\chi}(\{n_k\})$ is given by (7.3.20), maximum in (7.5.14), (7.5.15) and (7.5.16) is taken over all increasing sequences $\{n_k\}$ of positive

<u>*integers and*</u> $\max\limits_{\{n_k\}} [\tilde{V}_H^{\delta}(\alpha,\beta,\{n_k\})]$ <u>*and*</u> $\max\limits_{\{n_k\}} [\bar{V}_H^{\delta}(\alpha,\beta,\{n_k\})]$ <u>*are finite.*</u>

<u>PROOF</u>. First, let $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, be an entire harmonic function with finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,H)$. Then, using Lemma 7.5.1, the relation (7.5.14) follows on applyir (1.3.11) and (1.3.12) to the function $h_\delta(z) = \sum\limits_{n=0}^{\infty} \Delta_{n,\delta}(H,R_0)(z/R_0$ while (7.5.15) and (7.5.16) follow on applying Theorem 6.3.2 to the function $h_\delta(z)$. This proves the necessity part of the theore

Conversely, if, for $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, $\max\limits_{\{n_k\}} [\tilde{V}_H^{\delta}(\alpha,\beta,\{n_k\})]$ is finite, then $\lim_{n\to\infty}(\Delta_{n,\delta}(H,R_0))^{1/n} = 0$ and so H is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

We note that the choice $\alpha(x) = \log x$ and $\beta(x) = (\exp x)^c$, $0 < c < \infty$, is not permissible in Theorems 7.5.1 to 7.5.3 and so the results obtained therein fail to give any specific information about the influence of the type and lower type, given by (1.11.4) and (1.11.5), of an entire harmonic function on its degree of approximation. Thus, we now obtain characterizations of type and lower type of an entire harmonic function H in terms of the approximation errors given by (4.4.3) and (4.4.4).

<u>THEOREM</u> 7.5.4. <u>*Let the harmonic function*</u> $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, <u>*and let the approximation errors*</u> $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, <u>*be given*</u>

by (4.4.3) and (4.4.4). Then, a necessary and sufficient condition that H has an analytic continuation as an entire harmonic function of order $\rho_\infty(H)$, $0 < \rho_\infty(H) < \infty$, and nonzero finite type $T_\infty(H)$ is that

$$W_H^\delta = \limsup_{n\to\infty} n(\Delta_{n,\delta}(H,R_0))^{\rho_\infty(H)/n}, \quad 1 \le \delta \le \infty,$$

satisfies $0 < W_H^\delta < \infty$. Further, $W_H^\delta = e\,\rho_\infty(H)\,T_\infty(H)\,R_0^{\rho_\infty(H)}$ also holds.

PROOF. Necessity part of the theorem follows from Lemma 7.5.1 on applying (1.2.8) to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_0)(z/R_0)$ Conversely, if, for $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, W_H^δ satisfies $0 < W_H^\delta < \infty$ then we have

$$\limsup_{n\to\infty} \frac{n \log n}{-\log \Delta_{n,\delta}(H,R_0)} = \rho_\infty(H)$$

and so, by Theorem 7.5.1, H is an entire harmonic function of order $\rho_\infty(H)$. Sufficiency part of the theorem now follows from the necessity part. This proves the theorem.

REMARK. For $p = 1$, Theorem 7.5.4 was obtained in [58].

THEOREM 7.5.5. *Suppose $H \in \bar{H}_{R_0}$, $0 < R_0 < \infty$, has an analytic continuation as an entire harmonic function having order $\rho_\infty(H)$, $0 < \rho_\infty(H) < \infty$, and lower type $t_\infty(H)$. Then, if $\Delta_{n,\delta}(H,R_0)/\Delta_{n+1,\delta}(H,R_0)$, where the approximation errors $\Delta_{n,\delta}(H,R_0)$, $1 \le \delta \le \infty$, are given by (4.4.3) and (4.4.4), is ultimately a nondecreasing function of n, we have*

$$e\,\rho_\infty(H)\; t_\infty(H)\; R_o^{\rho_\infty(H)} = \lim_{n\to\infty} \inf n(\Delta_{n,\delta}(H,R_o))^{\rho_\infty(H)/n}.$$

PROOF. The theorem follows from Lemma 7.5.1 on applying (1.2.1 to the function $h_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(H,R_o)(z/R_o)^n$.

7.6. A generalized axisymmetric potential (GASP) G having the following ultra-spherical harmonic expansion (Section 1.12)

$$(7.6.1) \qquad G(x,y) \equiv G(r,\theta) = \sum_{n=0}^{\infty} b_n r^n C_n^u(\cos\theta),$$

where $x = r\cos\theta$, $y = r\sin\theta$, C_n^u are Gegenbauer polynomials (Section 1.12) and u is the positive constant occuring in the partial differential equation (5.1.1), is said to be entire, if the series on the right hand side of (7.6.1) converges uniformly on compact subsets of the whole plane.

Let $\bar{G}_{R_o}$, $0 < R_o < \infty$, as in Section 5.1, denote the class of all GASP's, regular on the closed disc $\bar{D}_{R_o}$ of radius R_o centered at the origin. For $G \in \bar{G}_{R_o}$, let the uniform norm $||\cdot||_{R_o,\infty}$ and L^δ-norm $||\cdot||_{R_o,\delta}$, $1 \le \delta < \infty$, be given by (5.1.3) and (5.1.4), respectively. Further, for $G \in \bar{G}_{R_o}$, let $\Delta_{n,\infty}(G,R_o)$ and $\Delta_{n,\delta}(G,R_o)$, $1 \le \delta < \infty$, the errors in approximating the GASP G by GASP polynomials of degree atmost n, respectively, in uniform norm and L^δ-norm be given by (5.1.5) and (5.1.6).

We note, from Theorem 5.2.1, that $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire GASP, if and only if,

$$(7.6.2) \qquad \lim_{n\to\infty} (\Delta_{n,\delta}(G,R_o))^{1/n} = 0,\; 1 \le \delta \le \infty.$$

In this section, we first introduce the concepts of generalized (α,β)-order and generalized lower (α,β)-order of an entire GASP. Then, we obtain necessary and sufficient condition in terms of the approximation errors $\Delta_{n,\delta}(G,R_o)$, $0 < R_o < \infty$, $1 \leq \delta \leq \infty$, given by (5.1.5) and (5.1.6), such that a GASP $G \in \bar{G}_{R_o}$ has an analytic continuation as an entire GASP having finite generalized (α,β)-order and finite generalized lower (α,β)-order. Finally, we obtain characterizations of type and lower type of an entire GASP, given by (1.12.6) and (1.12.7), in terms of the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, $0 < R_o < \infty$. Our results in this section generalize some results of McCoy [67].

We first give the following definition

DEFINITION 7.6.1. *An entire GASP G is said to be of generalized (α,β)-order $\rho_\infty(\alpha,\beta,G)$ and generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,G)$ if*

$$\begin{matrix}\rho_\infty(\alpha,\beta,G)\\ \lambda_\infty(\alpha,\beta,G)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,G))}{\beta(\log r)}$$

where $\alpha(x) \in \Lambda_*$, $\beta(x) \in \overset{o}{L}_*$ and

$$M(r,G) = \max_\theta |G(r,\theta)|.$$

With $\alpha(x) = \log x$ and $\beta(x) = x$ in Definition 7.6.1, $\rho_\infty(\alpha,\beta,G)$ becomes the order $\rho_\infty(G)$ of an entire GASP G.

We shall assume in this section, to avoid some trivial cases, that $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, is not a GASP polynomial.

The following lemma is needed in the proofs of Theorems 7.6.1 to 7.6.5.

LEMMA 7.6.1. *Let the GASP* $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, *has an analytic continuation as an entire GASP and let* $1 \le \delta \le \infty$. *Then, the function* $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, *where the approximation errors* $\Delta_{n,\delta}(G,R_o)$, $1 \le \delta \le \infty$, *are given by* (5.1.5) *and* (5.1.6), *is entire. Further,*

(i) *For* $\alpha(x) \in \Lambda_*$ *and* $\beta(x) \in L^o_*$,

$$\lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,G))}{\beta(\log r)} = \lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,g_\delta))}{\beta(\log r)}$$

(ii) *If G is of order* $\rho_\infty(G)$, $0 < \rho_\infty(G) < \infty$, *type* $T_\infty(G)$ *and lower type* $t_\infty(G)$, *then* $g_\delta(z)$ *is also of order* $\rho_\infty(G)$, *type* $T_\infty(G)$ *and lower type* $t_\infty(G)$.

PROOF. (i) Since G is entire, by (7.6.2), it follows that $g_\delta(z)$ is also entire. Now, since $n^{1/n} \to 1$ as $n \to \infty$, it follows, from (5.2.16), that, given $r' > 1$, we have

$$\Delta_{n,\delta}(G,R_o) \le K^o_\delta\, K\, M(r,G)(R_o\, r'/r)^{n+1}$$

for all sufficiently large values of r and n, where K^o_δ and K are constants. This gives that

$$\mu(r/r', g_\delta) \le K^o_\delta\, K\, M(r,G) \tag{7.6.3}$$

for all sufficiently large values of r. For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L^o_*$, the above relation, on using (1.4.8), gives that

$$\lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,g_\delta))}{\beta(\log r)} \le \lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,G))}{\beta(\log r)} . \tag{7.6.4}$$

Now, since for any positive constant c,

$$(\Gamma(n+2c)(n+2)(n+c)/\Gamma(n+1))^{1/n} \to 1$$

as $n \to \infty$, given $r_* > 1$ there exists a constant $\tilde{K}_{r_*}$ such that, for $n \geq 1$, we have

$$\Gamma(n+2c)(n+2)(n+c)/\Gamma(n+1) < \tilde{K}_{r_*} r_*^n.$$

Thus, from (5.2.28) we get

$$(7.6.5)\quad M(r,G) \leq |b_o| + \frac{K_\delta\ K^*\ \tilde{K}_{r_*}}{\Gamma(2u)} \sum_{n=1}^{\infty} \Delta_{n-1,\delta}(G,R_o)(rr_*/R_o)^n$$
$$= |b_o| + \frac{K_\delta\ K^*\ \tilde{K}_{r_*}}{\Gamma(2u)}(rr_*/R_o)\ M(rr_*,g_\delta),$$

where K_δ and K^* and $|b_o|$ are constants and u is the positive constant occuring in the equation (5.1.1). The above relation easily gives that

$$(7.6.6)\quad \lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,G))}{\beta(\log r)} \leq \lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,g_\delta))}{\beta(\log r)}.$$

Combining (7.6.4) and (7.6.6) we get Part (i) of the lemma.

(ii) If the entire GASP G is of order $\rho_\infty(G)$, then it follows, from part (i) of the lemma, that $g_\delta(z)$, $1 \leq \delta \leq \infty$, is an entire function of order $\rho_\infty(G)$. Now, if $0 < \rho_\infty(G) < \infty$, let G be of type $T_\infty(G)$ and lower type $t_\infty(G)$. From (7.6.3), on using (1.4.7), we get

$$(7.6.7)\qquad T_\infty(g_\delta) \leq T_\infty(G),\ t_\infty(g_\delta) \leq t_\infty(G),$$

since $r' > 1$ is arbitrary. Here $T_\infty(g_\delta)$ and $t_\infty(g_\delta)$ are, respectively, type and lower type of $g_\delta(z)$. Now, since $r_* > 1$ is

arbitrary, from (7.6.5), we obtain

(7.6.8) $\quad T_\infty(G) \le T_\infty(g_\delta),\ t_\infty(G) \le t_\infty(g_\delta)$.

Part (ii) of the lemma now follows from (7.6.7) and (7.6.8).

This proves the lemma.

We now have

THEOREM 7.6.1. *Let the* GASP $G \in \bar{G}_{R_0}$, $0 < R_0 < \infty$ *and let the approximation errors* $\Delta_{n,\delta}(G,R_0)$, $1 \le \delta \le \infty$, *be given by* (5.1.5) *and* (5.1.6). *A necessary and sufficient condition that* G *has an analytic continuation as an entire* GASP *having finite generalized* (α,β)-*order* $\rho_\infty(\alpha,\beta,G)$ *is that*

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.6), *then*

(7.6.9) $\quad \rho_\infty(\alpha,\beta,G) = U_G^\delta(\alpha,\beta)$

(ii) *If* $\alpha = \beta$ *and* $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$, *then*

(7.6.10) $\quad \rho_\infty(\alpha,\alpha,G) = Q(U_G^\delta(\alpha,\alpha))$,

where

(7.6.11) $$U_G^\delta(\alpha,\beta) = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(\frac{1}{n}\log\left(\Delta_{n,\delta}(G,R_0)\right)^{-1}\right)} < \infty.$$

PROOF. First, let $G \in \bar{G}_{R_0}$, $0 < R_0 < \infty$, has an analytic continuation as an entire GASP having generalized (α,β)-order $\rho_\infty(\alpha,\beta,G)$. Then, by Lemma 7.6.1, the relation (7.6.9) follows on applying (1.3.7) to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_0)(z/R_0)^n$, $1 \le \delta \le \infty$, while the relation (7.6.10) follows on applying

Theorem 6.2.1 to the function $g_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, (7.6.11) holds, then $\lim_{n \to \infty}(\Delta_{n,\delta}(G,R_o))^{1/n} = 0$ and so, by (7.6.2), G is entire Sufficiency part of the theorem now follows from the necessity part.

This completes the proof of the theorem.

REMARK. For $R_o = 1$, $\delta = \infty, \alpha(x) = \log x$ and $\beta(x) = x$, the above theorem was obtained by McCoy [67], using a different method.

THEOREM 7.6.2. *Let the GASP $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$ and let the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, be given by (5.1.5) and (5.1.6). Assume that $\Delta_{n,\delta}(G,R_o)/\Delta_{n+1,\delta}(G,R_o)$ is ultimately a nondecreasing function of n. A necessary and sufficient condition that G has an analytic continuation as an entire GASP having finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,G)$ is that*

(i) *If $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9), then*

$$\lambda_\infty(\alpha,\beta,G) = V_G^\delta(\alpha,\beta). \tag{7.6.12}$$

(ii) *If $\alpha \equiv \beta$ and $\alpha(x)$ belongs to Ω or $\bar{\Omega}$, then*

$$\lambda_\infty(\alpha,\alpha,G) = Q(V_G^\delta(\alpha,\alpha)), \tag{7.6.13}$$

where

$$V_G^\delta(\alpha,\beta) = \liminf_{n \to \infty} \frac{\alpha(n)}{\beta(\frac{1}{n}\log(\Delta_{n,\delta}(G,R_o))^{-1})} < \infty. \tag{7.6.14}$$

PROOF. First, let $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, be an entire GASP of generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,G)$ and let

$\Delta_{n,\delta}(G,R_o)/\Delta_{n+1,\delta}(G,R_o)$ be ultimately a nondecreasing function of n. Then, using Lemma 7.6.1, the relation (7.6.12) follows on applying (1.3.10) to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$ while (7.6.13) follows on applying Theorem 6.3.1 to the function $g_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $G \in \bar{G}_{R_o}$, $\Delta_{n,\delta}(G,R_o)/\Delta_{n+1,\delta}(G,R_o)$ is ultimately nondecreasing and (7.6.14) holds, then $\lim_{n \to \infty} (\Delta_{n,\delta}(G,R_o))^{1/n} = 0$ and so G is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

THEOREM 7.6.3. Let the GASP $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, and let the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, be given by (5.1.5) and (5.1.6). A necessary and sufficient condition that G has an analytic continuation as an entire GASP having finite generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,G)$ is that

(i) If $\alpha(x)$ and $\beta(x)$ satisfy (1.3.8) and (1.3.9), then

$$(7.6.15)\quad \lambda_\infty(\alpha,\beta,G) = \max_{\{n_k\}} [\tilde{V}_G^\delta(\alpha,\beta,\{n_k\})] = \max_{\{n_k\}} [\bar{V}_G^\delta(\alpha,\beta,\{n_k\})] .$$

(ii) If $\alpha \equiv \beta$ and $\alpha(x) \in \Omega$, then

$$(7.6.16)\quad \lambda_\infty(\alpha,\alpha,G) = \max_{\{n_k\}} [Q(\tilde{V}_G^\delta(\alpha,\alpha,\{n_k\}))]$$

$$= \max_{\{n_k\}} [Q(\bar{V}_G^\delta(\alpha,\alpha,\{n_k\}))] .$$

(iii) If $\alpha \equiv \beta$, $\alpha(x) \in \bar{\Omega}$ and the principal indices $\{n_m\}$ of the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$ satisfy $\alpha(n_m) \sim \alpha(n_{m-1})$ as

$m \to \infty$, *then*

$$(7.6.17)\quad \lambda_\infty(\alpha,\alpha,G) = \max_{\{n_k\}} \left[Q_{\tilde{\chi}(\{n_k\})}(\tilde{V}_G^\delta(\alpha,\alpha,\{n_k\}))\right]$$

$$= \max_{\{n_k\}} \left[Q_{\tilde{\chi}(\{n_k\})}(\bar{V}_G^\delta(\alpha,\alpha,\{n_k\}))\right],$$

where

$$\tilde{V}_G^\delta(\alpha,\beta,\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k}\log(\Delta_{n_k,\delta}(G,R_o))^{-1})},$$

$$\bar{V}_G^\delta(\alpha,\beta,\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{n_k-n_{k-1}}\log(\Delta_{n_{k-1},\delta}(G,R_o)/\Delta_{n_k,\delta}(G,R_o)))}$$

$\tilde{\chi}(\{n_k\})$ *is given by* (7.3.20), *maximum in* (7.6.15), (7.6.16) *and* (7.6.17) *is taken over all increasing sequences* $\{n_k\}$ *of positive integers and* $\max_{\{n_k\}}[\tilde{V}_G^\delta(\alpha,\beta,\{n_k\})]$ *and* $\max_{\{n_k\}}[\bar{V}_G^\delta(\alpha,\beta,\{n_k\})]$ *are finite*.

PROOF. First, let $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, be an entire GASP with generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,G)$. Then, using Lemma 7.6.1, the relation (7.6.15) follows on applying (1.3.11) and (1.3.12) to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$, while (7.6.16) and (7.6.17) follow on applying Theorem 6.3.2 to the function $g_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, $\max_{\{n_k\}}[\tilde{V}_G^\delta(\alpha,\beta,\{n_k\})]$ is finite, then $\lim_{n\to\infty}(\Delta_{n,\delta}(G,R_o))^{1/n} = 0$ and so G is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

The choice $\alpha(x) = \log x$ and $\beta(x) = (\exp x)^c$, $0 < c < \infty$, is not permissible in Theorems 7.6.1 to 7.6.3 and so the results obtained therein do not give any specific information about the influence of the type and lower type, given by (1.12.6) and (1.12.7), of an entire GASP on its degree of approximation. Thus, we now obtain characterizations of type and lower type of an entire GASP in terms of the approximations errors $\Delta_{n,\delta}(G,R_o)$, $0 < R_o < \infty$, $1 \leq \delta \leq \infty$, given by (5.1.5) and (5.1.6).

THEOREM 7.6.4. *Let the GASP $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, and let the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, be given by (5.1.5) and (5.1.6). A necessary and sufficient condition that G has an analytic continuation as an entire GASP of order $\rho_\infty(G)$, $0 < \rho_\infty(G) < \infty$, and nonzero finite type $T_\infty(G)$ is that*

$$(7.6.18) \qquad W_G^\delta = \limsup_{n \to \infty} n(\Delta_{n,\delta}(G,R_o))^{\rho_\infty(G)/n}, \quad 1 \leq \delta \leq \infty,$$

satisfies $0 < W_G^\delta < \infty$. Further, $W_G^\delta = e\,\rho_\infty(G)T_\infty(G)R_o^{\rho_\infty(G)}$ also holds.

PROOF. Necessity part of the theorem follows from Lemma 7.6.1 on applying (1.2.8) to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$. Conversely, if, for $G \in \bar{G}_{R_o}$, $0 < R_o < \infty$, W_G^δ satisfies $0 < W_G^\delta < \infty$, then we have

$$\limsup_{n \to \infty} \frac{n \log n}{-\log \Delta_{n,\delta}(G,R_o)} = \rho_\infty(G)$$

and so, by Theorem 7.6.1, G is an entire GASP of order $\rho_\infty(G)$. Sufficiency part of the theorem now follows from the necessity part. This proves the theorem.

REMARK. The above theorem improves a result of McCoy [67] for the case $\delta = \infty$ and $R_o = 1$. McCoy proved that a GASP $G \varepsilon \bar{G}_1$ is an entire GASP of nonzero finite order $\rho_\infty(G)$ and finite type if and only if W_G^∞, as given by (7.6.18), is finite.

THEOREM 7.6.5. Suppose $G \varepsilon \bar{G}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire GASP having order $\rho_\infty(G)$, $0 < \rho_\infty(G) < \infty$, and lower type $t_\infty(G)$. If $\Delta_{n,\delta}(G,R_o)/\Delta_{n+1,\delta}(G,R_o)$, where the approximation errors $\Delta_{n,\delta}(G,R_o)$, $1 \leq \delta \leq \infty$, are given by (5.1.5) and (5.1.6), is ultimately a nondecreasing function of n, then

$$e\, \rho_\infty(G)\ t_\infty(G)\ R_o^{\rho_\infty(G)} = \liminf_{n \to \infty} n(\Delta_{n,\delta}(G,R_o))^{\rho_\infty(G)/n}.$$

PROOF. The theorem follows from Lemma 7.6.1 on applying (1.2.15) to the function $g_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(G,R_o)(z/R_o)^n$.

7.7. A generalized biaxisymmetric potential (GBSP) F having the following expansion (Section 1.13)

$$(7.7.1)\qquad F(x,y) \equiv F(r,\theta) = \sum_{n=0}^{\infty} c_n\, r^{2n}\, P_n^{(u,v)}(\cos 2\theta),\quad u,v > -1/2,$$

where $x = r\cos\theta$, $y = r\sin\theta$, $P_n^{(u,v)}$ are Jacobi polynomials and u and v are the constants occuring in the partial differential equation (5.4.1), is said to be entire if the series on the right hand side of (7.7.1) converges uniformly on the compact subsets of the whole plane.

Let $\bar{F}_{R_o}$, $0 < R_o < \infty$, as in Section 5.4, be the class of all GBSP's regular on the closed disc $\bar{D}_{R_o}$ of radius R_o centered

at the origin. For $F \in \bar{F}_{R_o}$, let the uniform norm $||\cdot||_{R_o,\infty}$ and L^δ-norm $||\cdot||_{R_o,\delta}$, $1 \leq \delta < \infty$, be given by (5.4.3) and (5.4.4), respectively. Further, for $F \in \bar{F}_{R_o}$, let $\Delta_{n,\infty}(F,R_o)$ and $\Delta_{n,\delta}(F,R_o)$, $1 \leq \delta < \infty$, the errors in approximating the GBSP F by GBSP polynomials of degree atmost 2n, respectively, in unifc norm and L^δ-norm be given by (5.4.5) and (5.4.6).

We note, from Theorem 5.5.1, that $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire GBSP, if and only if,

(7.7.2) $$\lim_{n \to \infty} (\Delta_{n,\delta}(F,R_o))^{1/2n} = 0.$$

In this section, we first introduce the concepts of generalized (α,β)-order and generalized lower (α,β)-order of an entire GBSP. We then obtain necessary and sufficient condition in terms of the approximation errors $\Delta_{n,\delta}(F,R_o)$, $0 < R_o < \infty$, $1 \leq \delta \leq \infty$, given by (5.4.5) and (5.4.6), such that a GBSP $F \in \bar{F}_{R_o}$ has an analytic continuation as an entire GBSP having finite generalized (α,β)-order and finite generalized lower (α,β)-orde Our results generalize some results of McCoy ([68], [70]).

We first have the following definition

DEFINITION 7.7.1. An entire GBSP F is said to be of generalized (α,β)-order $\rho_\infty(\alpha,\beta,F)$ and generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,F)$ if

$$\begin{matrix}\rho_\infty(\alpha,\beta,F) \\ \lambda_\infty(\alpha,\beta,F)\end{matrix} = \lim_{r \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\log M(r,F))}{\beta(\log r)},$$

where $\alpha(x) \in \Lambda_*$, $\beta(x) \in L_*^0$ *and*

$$M(r,F) = \max_{0 \le \theta \le 2\pi} |F(r,\theta)|.$$

To avoid some trivial cases, in this section, we shall assume that $F \in \bar{F}_{R_0}$, $0 < R_0 < \infty$, is not a GBSP polynomial.

The following lemma is a key result in the proofs of Theorems 7.7.1 to 7.7.3.

LEMMA 7.7.1. *Let the* GBSP $F \in \bar{F}_{R_0}$, $0 < R_0 < \infty$, *has an analytic continuation as an entire* GBSP *and let* $1 \le \delta \le \infty$. *Then, the function* $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_0)(z/R_0)^{2n}$, *where the approximation errors* $\Delta_{n,\delta}(F,R_0)$, $1 \le \delta \le \infty$, *are given by* (5.4.5) *and* (5.4.6), *is entire. Further, for* $\alpha(x) \in \Lambda_*$ *and* $\beta(x) \in L_*^0$,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,F))}{\beta(\log r)} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log M(r,f_\delta))}{\beta(\log r)}.$$

PROOF. Since the GBSP F is entire, by (7.7.2), we obtain that $f_\delta(z)$ is entire. Now, since $n^{1/n} \to 1$ as $n \to \infty$, it follows, from (5.5.15), that, given $r' > 1$, we have

$$\Delta_{n,\delta}(F,R_0) \le \tilde{K}_\delta^0 \tilde{K} M(r,F) (R_0 r'/r)^{2(n+1)}$$

for all sufficiently large values of r and n, where $\tilde{K}_\delta^0$ and $\tilde{K}$ are constants. This gives that

$$\mu(r/r', f_\delta) \le \tilde{K}_\delta^0 \tilde{K} M(r,F)$$

for all sufficiently large values of r. For $\alpha(x) \in \Lambda_*$ and $\beta(x) \in L_*^0$, the above inequality, on using (1.4.8), gives that

$$(7.7.3)\quad \lim_{r\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\alpha(\log M(r,f_\delta))}{\beta(\log r)} \le \lim_{r\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\alpha(\log M(r,F))}{\beta(\log r)}\ .$$

For $u > -1/2$ and $v > -1/2$, let $A(n,u,v)$ be given by (5.5.5). If $\gamma = \max(u,v)$, then

$$((n+1)(2n+u+v+1)(\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)})^2\, A(n,u,v))^{1/n} \to 1 \text{ as } n \to \infty.$$

Hence, given $r_* > 1$ there exists a constant $\bar{K}_{r_*}$ such that, for $n \ge 1$, we have

$$(n+1)(2n+u+v+1)\ (\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)})^2\, A(n,u,v) < \bar{K}_{r_*}\ r_*^{2n}.$$

Thus, from (5.5.24), we have

$$\begin{aligned} M(r,F) &\le |c_o| + \frac{\tilde{K}_\delta\ \bar{K}_{r_*}}{\Gamma(\gamma+1)} \sum_{n=1}^{\infty} \Delta_{n-1,\delta}(F,R_o)(rr_*/R_o)^{2n} \\ &= |c_o| + \frac{\tilde{K}_\delta\ \bar{K}_{r_*}}{\Gamma(\gamma+1)}\ (rr_*/R_o)^2\ M(rr_*,f_\delta) \end{aligned}$$

where $\tilde{K}_\delta$ and $|c_o|$ are constants. This inequality easily gives that

$$(7.7.4)\quad \lim_{r\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\alpha(\log M(r,F))}{\beta(\log r)} \le \lim_{r\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\alpha(\log M(r,f_\delta))}{\beta(\log r)}\ .$$

The lemma now follows on combining (7.7.3) and (7.7.4).

We now have

THEOREM 7.7.1. Let the GBSP $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, and let the approximation errors $\Delta_{n,\delta}(F,R_o)$, $1 \le \delta \le \infty$, be given by (5.4.5) and (5.4.6). A necessary and sufficient condition that F has an analytic continuation as an entire GBSP having finite generalized (α,β)-order $\rho_\infty(\alpha,\beta,F)$ is that

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.6), *then*

(7.7.5) $$\rho_\infty(\alpha,\beta,F) = U_F^\delta(\alpha,\beta)$$

(ii) *If* $\alpha \equiv \beta$ *and* $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$, *then*

(7.7.6) $$\rho_\infty(\alpha,\alpha,F) = Q(U_F^\delta(\alpha,\alpha)),$$

where

(7.7.7) $$U_F^\delta(\alpha,\beta) = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(\frac{1}{2n}\log(\Delta_{n,\delta}(F,R_o))^{-1})} < \infty.$$

PROOF. First let $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, has an analytic continuation as an entire GBSP having generalized (α,β)-order $\rho_\infty(\alpha,\beta,F)$. Then, by Lemma 7.7.1, the relation (7.7.5) follows on applying (1.3.7) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}, 1 \le \delta \le \infty$, while the relation (7.7.6) follows on applying Theorem 6.2.1 to the function $f_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, (7.7.7) holds then $\lim_{n\to\infty}(\Delta_{n,\delta}(F,R_o))^{1/2n} = 0$ and so, by (7.7.2), F is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

REMARK. For $R_o = 1$, $\delta = \infty$, $\alpha(x) = \log_p x$, $p \ge 1$ and $\beta(x) = x$, Theorem 7.7.1 gives the correct version of a result of McCoy [68]. For $R_o = 1$, $1 \le \delta \le \infty$, $\alpha(x) = \log x$ and $\beta(x) = x$, the above

theorem gives the correct version of another result of McCoy [70].

THEOREM 7.7.2. *Let the* GBSP $F \in \bar{F}_{R_0}$, $0 < R_0 < \infty$ *and let the approximation errors* $\Delta_{n,\delta}(F,R_0)$, $1 \leq \delta \leq \infty$, *be given by* (5.4.5) *and* (5.4.6). *Assume that* $\Delta_{n,\delta}(F,R_0)/\Delta_{n+1,\delta}(F,R_0)$ *is ultimately a nondecreasing function of* n. *A necessary and sufficient condition that* F *has an analytic continuation as an entire* GBSP *having finite generalized lower* (α,β)-*order* $\lambda_\infty(\alpha,\beta,F)$ *is that*

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.8) *and* (1.3.9), *then*

$$\lambda_\infty(\alpha,\beta,F) = V_F^\delta(\alpha,\beta) \tag{7.7.8}$$

(ii) *If* $\alpha \equiv \beta$ *and* $\alpha(x)$ *belongs to* Ω *or* $\bar{\Omega}$, *then*

$$\lambda_\infty(\alpha,\alpha,F) = Q(V_F^\delta(\alpha,\alpha)), \tag{7.7.9}$$

where

$$V_F^\delta(\alpha,\beta) = \liminf_{n\to\infty} \frac{\alpha(n)}{\beta(\frac{1}{2n}\log(\Delta_{n,\delta}(F,R_0))^{-1})} < \infty. \tag{7.7.10}$$

PROOF. First, let $F \in \bar{F}_{R_0}$, $0 < R_0 < \infty$, be an entire GBSP of generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,F)$ and let $\Delta_{n,\delta}(F,R_0)/\Delta_{n+1,\delta}(F,R_0)$ be ultimately nondecreasing. Then, using Lemma 7.7.1, the relation (7.7.8) follows on applying (1.3.10) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_0)(z/R_0)^{2n}$, while (7.7.9) follows on applying Theorem 6.3.1 to the function $f_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $F \in \bar{F}_{R_0}$, $\Delta_{n,\delta}(F,R_0)/\Delta_{n+1,\delta}(F,R_0)$ is ultimately nondecreasing and (7.7.10) holds, then

$\lim_{n \to \infty}(\Delta_{n,\delta}(F,R_o))^{1/2n} = 0$ and so F is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

THEOREM 7.7.3. *Let the GBSP* $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, *and let the approximation errors* $\Delta_{n,\delta}(F,R_o)$, $1 \le \delta \le \infty$, *be given by* (5.4.5) *and* (5.4.6). *A necessary and sufficient condition that F has an analytic continuation as an entire GBSP having finite generalized lower* (α,β)*-order* $\lambda_\infty(\alpha,\beta,F)$ *is that*

(i) *If* $\alpha(x)$ *and* $\beta(x)$ *satisfy* (1.3.8) *and* (1.3.9), *then*

$$\lambda_\infty(\alpha,\beta,F) = \max_{\{n_k\}} [\tilde{V}_F^\delta(\alpha,\beta,\{n_k\})] = \max_{\{n_k\}} [\bar{V}_F^\delta(\alpha,\beta,\{n_k\})] \, . \tag{7.7.11}$$

(ii) *If* $\alpha \equiv \beta$ *and* $\alpha(x) \in \Omega$, *then*

$$\lambda_\infty(\alpha,\alpha,F) = \max_{\{n_k\}} [Q(\tilde{V}_F^\delta(\alpha,\alpha,\{n_k\}))] \tag{7.7.12}$$

$$= \max_{\{n_k\}} [Q(\bar{V}_F^\delta(\alpha,\alpha,\{n_k\}))] \, .$$

(iii) *If* $\alpha \equiv \beta$, $\alpha(x) \in \bar{\Omega}$ *and the principal indices* $\{2n_m\}$ *of the function* $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}$ *satisfy* $\alpha(n_m) \sim \alpha(n_{m+1})$ *as* $m \to \infty$, *then*

$$\lambda_\infty(\alpha,\alpha,F) = \max_{\{n_k\}} [Q_{\tilde{\chi}(\{n_k\})}(\tilde{V}_F^\delta(\alpha,\alpha,\{n_k\}))] \tag{7.7.13}$$

$$= \max_{\{n_k\}} [Q_{\bar{\chi}(\{n_k\})}(\bar{V}_F^\delta(\alpha,\alpha,\{n_k\}))]$$

where,

$$\tilde{V}_F^{\delta}(\alpha,\beta,\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{2n_k}\log(\Delta_{n_k,\delta}(F,R_o))^{-1})},$$

$$\bar{V}_F^{\delta}(\alpha,\beta,\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\beta(\frac{1}{2(n_k-n_{k-1})}\log(\Delta_{n_{k-1},\delta}(F,R_o)/\Delta_{n_k,\delta}(F,R_o)))},$$

$\tilde{\chi}(\{n_k\})$ <u>is given by</u> (7.3.20), <u>maximum in</u> (7.7.11),(7.7.12) <u>and</u> (7.7.13) <u>is taken over all increasing sequences</u> $\{n_k\}$ <u>of positive integers and</u> $\max_{\{n_k\}} [\tilde{V}_F^{\delta}(\alpha,\beta,\{n_k\})]$ <u>and</u> $\max_{\{n_k\}} [\bar{V}_F^{\delta}(\alpha,\beta,\{n_k\})]$ <u>are finite</u>.

<u>PROOF</u>. First, let $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, be an entire GBSP with generalized lower (α,β)-order $\lambda_\infty(\alpha,\beta,F)$. Then, using Lemma 7.7.1, the relation (7.7.11) follows on applying (1.3.11) and (1.3.12) to the function $f_\delta(z) = \sum_{n=0}^{\infty} \Delta_{n,\delta}(F,R_o)(z/R_o)^{2n}$, while (7.7.12) and (7.7.13) follow on applying Theorem 6.3.2 to the function $f_\delta(z)$. This proves the necessity part of the theorem.

Conversely, if, for $F \in \bar{F}_{R_o}$, $0 < R_o < \infty$, $\max_{\{n_k\}} [\tilde{V}_F^{\delta}(\alpha,\beta,\{n_k\})]$ is finite, then $\lim_{n\to\infty}(\Delta_{n,\delta}(F,R_o))^{1/2n} = 0$ and so F is entire. Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

LIST OF RESEARCH PAPERS

1. Polynomial approximation of an entire function of slow growth (Chapters 6 and 7).
J. Approximation Theory 32 (1981), 64-75.

2. On the approximation of an analytic function in L^{α}-norm (Chapter 1).
Tamkang J. Math. 12 (1981), 67-76.

3. Approximation of entire functions over Caratheodory domains (Chapter 7).
Bull. Austral. Math. Soc. (Accepted for publication).

4. On the coefficients of a function analytic in the unit disc having slow rate of growth (Chapter 3).
Annali di Matematica pura ed applicata (Accepted for publication).

5. Approximation of entire harmonic functions in R^3 (Chapter 7).
Indian J. Pure Appl. Math. (Accepted for publication).

6. On the approximation and interpolation of an analytic function (Chapter 1).
Math. Student (Accepted for publication).

7. On the coefficients of an entire function (Chapter 1).
Pure and Appl. Math. Sciences (Accepted for publication).

8. On the growth of a function analytic in the unit disc (Chapter 2).
Communicated for publication.

9. On the growth of harmonic functions in R^3 (Chapter 4).
Communicated for publication.

10. On the approximation of generalized axisymmetric potentials (Chapter 5).
Communicated for publication.

11. On the growth and approximation of generalized biaxi-symmetric potentials (Chapter 5).
Communicated for publication.

12. On the generalized orders of an entire function of slow growth (Chapter 6).
Communicated for publication.

13. On the approximation of an entire harmonic function in R^{p+2} (Chapter 7).
Communicated for publication.

REFERENCES

1. Aborn, B.J. and Shankar, H. : Generalized growth for functions analytic in a finite disc. Pure and Appl.Math. Sciences 12 (1980), 83-94.

2. Armitage, D.H. : On the derivatives at the origin of entire harmonic functions. Glasgow Math. J. 20(1979), 147-154.

3. Bajpai, S.K. : Decomposition theorems for entire functions of Šeremeta growth. Glasnik Matematicki 11 (31) (1976), 11-18.

4. ________, Gautam, S.K.S. and Bajpai, S.S. : Generalizations of growth constants I. Ann. Polon. Math. 37 (1980), 13-24.

5. ________, Kapoor, G.P. and Juneja, O.P. : On entire functions of fast growth. Trans. Amer. Math. Soc. 203 (1975), 275-297.

6. ________ and Shah, S.M. : Approximation and interpolation of entire functions and generalized orders. Ann. Polon. Math. 37 (1980), 157-166.

7. ________ and Tanne, J. : Analogues of entire functions inequalities for an analytic function. Canad. J. Math. 27(2), (1975), 286-293.

8. ________, Tanne, J. and Whittier, D. : A decomposition theorem for an analytic function. J. Math. Anal. Appl. 48(1974), 736-742.

9. Basinger, R.C. : On the coefficients of an entire series with gaps. J. Math. Anal. Appl. 38 (1972), 790-792.

10. Batyrev, A.V. : On the question of best approximation of analytic functions by polynomials. Dokl. Akad. Nauk. USSR 76 (1951), 173-175.

11. Bergman, S. : Integral operators in the theory of linear partial differential equations.
Springer-Verlag, New York, 1969.

12. Bernstein, S.N. : Leçons sur les propriétés entrémales et al meilleure approximation des fonctions analytiques d'une variable réele.
Gauthier-Villars, Paris, 1926.

13. Beuermann, F. : Wachstumsordnung koeffizientenvachstum und nullsteluendichte bei potenzreihen mit endlichen konvergenzkus.
Math. Z. 33 (1931), 98-108.

14. Blumenthal : Sur le mode de croissance des fonctions entieres.
Bull. Société Math., 1907.

15. Boas, R.P., Jr. : Entire functions.
Academic Press, 1954.

16. Bogda, R.A. and Shankar, H. : The maximum term of a power series.
Math. Japon. 19 (1974), 195-203.

17. ________________ : Convolutions and growth numbers of analytic functions.
Rocky Mountain J. Math. 10(1980), 475-483.

18. Calderon, A.P. and Zygmund, A. : On higher gradients of harmonic functions.
Studia Math. 24 (1964), 211-226.

19. Clunie, J. : Note on integral functions of infinite order.
Quart. J. Math. (2), 6 (1955), 88-90.

20. ___________ : The behaviour of integral functions determined from their Taylor series.
Quart. J. Math. (2), 7 (1956), 175-182.

21. ___________ and Hayman, W.K. : The maximum term of a power series 1.
J. Analyse Math. 12 (1964), 143-186.

22. ___________ and Hayman, W.K. : The maximum term of a power series 2.
J. Analyse Math. 14 (1965), 15-65.

23. Courant, R. and Hilbert, D. : Methods of mathematical physics, Vol. II. Interscience Publishers, 1962.

24. Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. : Higher Transcendental functions; Vol. II. McGraw Hill, New York, 1953.

25. Fryant, A.J. : Growth and complete sequences of generalized axisymmetric potentials. J. Approximation Theory, 19 (1977), 361-370.

26. ______________ : Interpolation and approximation of generalized axisymmetric potentials. Siam J. Math. Anal. 9 (1978), 906-914.

27. ______________ : Growth of entire harmonic functions in R^3. J. Math. Anal. Appl. 66 (1978), 599-605.

28. ______________ : Interpolation and approximation of axisymmetric harmonic functions in R^3 Amer. J. Math. 100 (1978), 205-216.

29. ______________ : Growth and complete sequences of generalized bi-axially symmetric potentials. J. Differential Equations 31 (1979), 155-164.

30. Fugard, T.B. : Growth of entire harmonic functions in R^n, $n \geq 2$. J. Math. Anal. Appl. 74 (1980), 286-291.

31. Gervais, R. and Rahman, Q.I. : An extension of Carlson's theorem for entire functions of exponential type. Trans. Amer. Math. Soc. 235 (1978), 387-394.

32. ______________ : An extension of Carlson's theorem for entire functions of exponential type II. J. Math. Anal. Appl. 69 (1979), 585-602.

33. Gilbert, R.P. : Some inequalities for generalized axially symmetric potentials with entire and meromorphic associates. Duke J. Math. 32 (1965), 239-246.

34. ______________ : Function theoretic methods in partial differential equations. Academic Press, New York, 1969.

35. Giroux, A. : Approximation of entire functions over bounded domains. J. Approximation Theory 28 (1980), 45-53.

36. Goluzin, G.M. : Geometric theory of functions of a complex variable. Amer. Math. Soc. Transl., Vol. 26, Providence, R.I., 1969.

37. Gopal, K. : A study in the growth properties and mean values of analytic functions, Thesis, Meerut University, Meerut, 1979.

38. Gray, A. and Shah, S.M. : Holomorphic functions with gap power series. Math. Z. 86 (1965), 375-394.

39. Ibragimov, I.I. and Šihaliev, N.I. : On the best polynomial approximation in a space of analytic functions. Soviet Math. Dokl. 17 (1976), 381-384.

40. Iyer, V.G. : A property of the maximum modulus of an integral function. J. Indian Math. Soc. 6 (1942), 69-80.

41. Juneja, O.P. : On the coefficients of an entire series of finite order. Arch. Math. (Basel) 21 (1970), 374-378.

42. ______________ : On the coefficients of an entire series. J. Analyse Math. 24 (1971), 395-401.

43. ______________ : Approximation of an entire function. J. Approximation Theory 11 (1974) 343-349.

44. Juneja, O.P. and Kapoor, G.P. : On the lower order of entire functions. J. London Math. Soc. (2) 5 (1972), 310-312.

45. Juneja, O.P., Kapoor, G.P. and Bajpai, S.K. : On the (p,q)-order and lower (p,q)-order of an entire function. J. Reine Angew. Math. 282 (1976), 53-67.

46. ——— ——— ——— : On the (p,q)-type and lower (p,q)-type of an entire function. J. Reine Angew. Math. 290 (1977), 180-190.

47. Juneja, O.P. and Rizvi, S.R.H. : Interpolation and approximation of entire functions. Proc. Royal Irish Acad. 79, A, (1979), 37-47.

48. Juneja, O.P. and Singh, P. : On the growth of an entire series with gaps. J. Math. Anal. Appl. 30 (1970), 330-334.

49. Kapoor, G.P. : A note on the proximate order of functions analytic in the unit disc. Istanbul Univ. Fen. Fak. Mec. Ser.A, 36 (1971), 35-40.

50. ——————— : A study in the growth properties and coefficients of analytic functions. Thesis, Indian Institute of Technology, Kanpur, 1972.

51. ——————— : On the lower order of functions analytic in the unit disc. Math. Japon 17 (1972), 49-54.

52. ——————— : On the proximate order and maximum term of analytic functions in the unit disc. Rev. Roum. Math. Pures Appl. 18 (1973), 1207-1215.

53. ——————— and Gopal, K. : On the coefficients of functions analytic in the unit disc having fast rates of growth. Annali di Mat. pura ed applicate (IV), Vol. CXXI (1979), 337-349.

54. Kapoor, G.P. and Gopal, K. : Decomposition theorems for analytic functions having slow rates of growth in a finite disc. J. Math. Anal. Appl. 74 (1980), 446-455.

55. ________________ : Coefficient characterizations for functions analytic in the unit disc having fast rate of growth. Bulletin Mathematique (To appear).

56. Kapoor, G.P. and Juneja, O.P. : On the lower order of functions analytic in the unit disc II. Indian J. Pure Appl. Math. 7 (1976), 241-248.

57. Kapoor, G.P. and Nautiyal, A. : Polynomial approximation of an entire function of slow growth, J. Approximation Theory 32 (1981), 64-75.

58. ________________ : Approximation of entire harmonic functions in R^3. Indian J. Pure Appl. Math. (To appear)

59. Kövari, T. : On the maximum modulus and maximum term of functions analytic in the unit disc. J. London Math. Soc. 41 (1966), 129-137.

60. Leja, F. : Teoria Funkeji analitycznych Warszawa (1957).

61. Levin, B. Ja. : Distribution of zeros of entire functions. Translations, Amer. Math. Soc., Vol. 5, Providence, R.I., 1964.

62. Linden, C.N. : The representation of regular functions. J. London Math. Soc. 39 (1964), 19-30.

63. Lo, C.Y. : A bound for entire harmonic functions of three variables. Quart. Appl. Math. 26 (1968), 451-455.

64. Lorentz, G.G. : Approximation of functions. Holt, Rinehart and Winston, New York, 1966.

65. MacLane, G.R. : Asymptotic values of holomorphic functions. Rice University Studies, Houston, 1963.

66. Markushevich, A.I. : Theory of functions of a complex variable, Vol. III. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

67. McCoy, P.A. : Polynomial approximation and growth of generalized axisymmetric potentials Canad. J. Math. 31 (1979), 49-59.

68. ____________ : Polynomial approximation of generalized biaxisymmetric potentials. J. Approximation Theory 25 (1979), 153-168.

69. ____________ : Approximation and harmonic continuation of axially symmetric potentials in E^3. Pacific J. Math. 81 (1979), 481-491.

70. ____________ : Best L^p-approximation of generalized biaxisymmetric potentials. Proc. Amer. Math. Soc. 79 (1980), 435-439.

71. Meinardus, G. : Approximation of functions: Theory and numerical methods. Springer-Verlag, 1967.

72. Nandan, K., Doherey, R.P. and Srivastava, R.S.L. : Proximate order of an entire function with index pair (p,q). Indian J. Pure Appl. Math. 11 (1980), 33-39.

73. ____________ : On the generalized type and generalized lower type of an entire function with index pair (p,q). Indian J. Pure Appl. Math. 11 (1980), 1424-1433.

74. Nautiyal, A. and Mishra, A.K. : On the coefficients of an entire function. Pure and Appl. Math. Sciences (To appear).

75. Nautiyal, A. and Rizvi, S.R.H. : On the approximation of an analytic function in L^{α}-norm. Tamkang J. Math. 12 (1981), 67-76.

76. Nevanlinna, R. : Analytic functions. Springer-Verlag, 1970.

77. Plessis, N. Du : An introduction to potential theory. Oliver and Boyd, Edinburgh, 1970.

78. Polya, G. and Szegö, S. : Problems and theorems in analysis, Vol. II. Springer-Verlag, 1972.

79. Rahman, Q.I. : On entire functions of infinite order Tôhoku Math. J. (2), 8 (1956), 165-16

80. ________ : On the coefficients of an entire series of finite order. Math. Student 25 (1957), 113-121.

81. ________ : On a theorem of Shah. Publ. Math. (Dubrecen) 5 (1957), 40-43.

82. ________ : Carlson's theorem for functions of exponential type. J. Analyse Math. 22 (1969), 201-219.

83. Reddy, A.R. : Approximation of an entire function. J. Approximation Theory 3 (1970), 128-137.

84. ________ : Best polynomial approximation to certain entire functions. J. Approximation Theory 5 (1972), 97-112.

85. ________ : A contribution to best approximation in L^2-norm. J. Approximation Theory 11 (1974), 110-117.

86. Rice, J.R. : The degree of convergence for entire functions. Duke Math. J. 38 (1971), 429-440.

87. Rizvi, S.R.H. : Some aspects of expansions, approximation and interpolation of analytic functions. Thesis, Indian Institute of Technology, Kanpur, 1978.

88. Rizvi, S.R.H. and Nautiyal, A. : On the approximation and interpolation of an analytic function. Math. Student (To appear).

89. Sato, D. : On the rate of growth of entire functions of fast growth. Bull. Amer. Math. Soc. 69 (1963), 411-414.

90. Schönhage, A. : Über das wachstum zusammengesetzter funktionen. Math. Z. 73 (1960), 22-44.

91. Seneta, E. : Regularly varying functions. Lecture notes in mathematics, 508, Springer-Verlag, 1976.

92. Šeremeta, M.N. : Connection between the growth of a function analytic in a disc and the moduli of the coefficients of its Taylor series (Ukrainian). Vîsnik L'viv. Derž Univ. Ser. Meh. Mat. 1965, vyp. 2, 101-110 (M.R. 35 # 4378).

93. ________ : On the connection between the growth of the maximum modulus of an entire function and the moduli of the coefficients of its power series expansion. Amer. Math. Soc. Transl. (2) 88 (1970) 291-301.

94. Sewell, W.E. : Degree of approximation by polynomials in the complex domain. Ann. of Math. Studies No. 9, Princeton University Press, Princeton, 1942.

95. Shah, S.M. : The maximum term of an entire series. Math. Student 10 (1942), 80-82.

96. ________ : A note on the lower order of integral functions. J. Indian Math. Soc. 9 (1945), 50-52.

97. ________ : On the lower order of integral functions. Bull. Amer. Math. Soc. 52 (1946), 1046-1052.

98. Shah, S.M. : The maximum term of an entire series III. Quart. J. Math. 19 (1948), 220-223.

99. ________ : The maximum term of an entire series IV. Quart. J. Math. Oxford Ser. (2), 1 (1950), 112-116.

100. ________ : The maximum term of an entire series VI. J. Indian Math. Soc. 14 (1950), 21-28.

101. ________ : On the coefficients of an entire series of finite order. J. London Math. Soc. 25 (1951), 45-46.

102. ________ : Polynomial approximation of an entire function and generalized orders. J. Approximation Theory 19 (1977), 315-324.

103. ________ and Ishaq, M. : On the maximum modulus and the coefficients of an entire series. J. Indian Math. Soc. 16 (1952), 177-182.

104. Siciak, J. : Some applications of the method of extremal points. Colloquium Math. 11 (1964), 209-250.

105. Singh, J.P. : Approximation of an entire function. Yokohama Math. J. 19 (1971), 105-108.

106. Singh, S.K. : On the maximum term and rank of an entire function. Acta Math. 94 (1955), 1-12.

107. Smirnov, V.I. : Course of higher mathematics V. Gostekhizdat (1959).

108. ________ and Lebedev, N.A. : Functions of a complex variable: Constructive theory. M.I.T. Press, Mass., U.S.A. (1968).

109. Sons, L.R. : Value distribution and power series with moderate gaps. Michigan Math. J. 13 (1966), 425-433.

110. Sons, L.R. : Regularity of growth and gaps. J. Math. Anal. Appl. 24 (1968), 296-306.

111. __________ : Zeros of power series with large gaps. Indiana Univ.Math. J. 22 (1972), 199-205.

112. __________ : Corrigendum to ''Regularity of growth and gaps''. J. Math. Anal. Appl. 58 (1977), 232.

113. __________ : Value distribution of canonical products in the unit disc. Math. Japon. 22 (1977), 27-38.

114. __________ : Zeros of functions of slow growth in the unit disc. Math. Japon, 24 (1979), 271-282.

115. Srivastava, G.S. : A study in the growth properties and mean values of entire and subharmonic functions. Thesis, Indian Institute of Technology, Kanpur, 1970.

116. __________ and Juneja, O.P. : The maximum term of a power series. J. Math. Anal. Appl. 81 (1981), 1-7.

117. Srivastav, R.P. : On the derivatives of integral functions. Ganita 7 (1956), 29-44.

118. Srivastava, R.S.L. : On the order of integral functions. Ganita 10 (1959), 23-30.

119. __________ : On the maximum term of an integral function. Acta Math. Acad. Sci. Hungar. 13 (1962), 275-280.

120. __________ and Singh, P. : On the λ-type of an integral function of irregular growth. Arch. Math. 17 (1966), 342-346.

121. Srivastava, S.N. : On the order and type of integral functions. Riv. Math. Univ. Parma (2), 2 (1961), 265-280

122. Szegö, G. : Orthogonal polynomials. Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, R.I., 1967.

123. Titchmarsh, E.C. : Theory of functions. Oxford, 1960.

124. Tsuji, M. : Canonical product for a meromorphic function in a unit circle. J. Math. Soc. Japan 8 (1956), 7-21.

125. ________ : Potential theory in modern function theory. Maruzen, Tokyo, 1959.

126. Valiron, G. : Lectures on the general theory of integral functions. Chelsea Publishing Co., 1949.

127. ________ : Fonctions analytiques. Paris Presses Universitaires de France, 1954.

128. Varga, R.S. : On an extension of a result of S.N. Bernstein. J. Approximation Theory 1 (1968), 176-179.

129. Walsh, J.L. : Interpolation and approximation by rational functions in the complex domain. Amer. Math. Soc., Providence, R.I., 1935.

130. Weinstein, A. : Generalized axisymmetric potential theory. Bull. Amer. Math. Soc. 59 (1953), 20-38.

131. Whittaker, J.M. : The lower order of integral functions. J. London Math. Soc. 8 (1933), 20-27.

132. Winiarski, T. : Approximation and interpolation of entire functions. Ann. Polon. Math. 23 (1970), 259-273.